Topic: Congruent triangles
Year group: Second year

Lesson Plan taught: February, 2016
At Coláiste Cholmáin, Claremorris, Second year class
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Title of the Lesson

"To Be or Not To Be: Congruent Triangles"

Brief description of the lesson

By drawing congruent triangles, students will notice the minimal number of conditions that determine when two triangles are congruent. The tasks will present the students with a challenge that requires thinking and presentation of ideas to their peers.

Aims of the lesson

a) **Short term aim:** We would like our students to establish what is the least amount of information required in order for two triangles to be considered congruent.
b) **Long term aims:** We would like our students:
   i. to gain confidence in dealing with abstract concepts,
   ii. to develop ideas involved in mathematical proof through the construction of counterexamples.
c) We would like to support our students in developing their literacy and numeracy skills through discussing ideas.

Learning Outcomes

As a result of studying this topic students will be able to:

a) decide which conditions are necessary in order to show that two triangles are congruent,
b) produce counterexamples to show that we cannot always draw congruent triangles using any three measurements (in particular, show that AAA and SSA are not sufficient conditions for congruence)
c) present logical ideas to their peers.

Background and Rationale

The concept of congruent triangles plays a significant role in both Junior and Senior Cycle mathematics as part of many abstract proofs (see Section B of the syllabus, Geometry for Post-primary School Mathematics, p.37-83). It is a challenging topic to teach effectively. The congruence axioms may seem dry and theoretical to students. In the past we have taught this topic using ideas from Junior Certificate Mathematics Guidelines for Teachers (DES 2002, Geometry Lesson Idea 14, page 72), where students would have constructed four triangles given three specific side lengths, two angle measures and a side length between them, the lengths of two sides and a measure of an included angle or a right angle, hypotenuse and one other side. The students would then have cut these out from a cardboard and place on a template made by a teacher to see that it is impossible to draw a triangle which is not identical to all the others with these measurements. The aim of that activity was to allow students to convince themselves of the truth of congruence conditions using concrete materials. In this lesson we intend to develop another approach to teaching this challenging topic.

Research

In preparation of this lesson plan the following materials have been used:

a) Junior Certificate Mathematics Guidelines for Teachers (DES 2002)
b) First and second year Teachers Handbooks (from the Project Maths website www.projectmaths.ie)
c) Congruent Shapes - selected excerpts from the Japanese teachers' manuals obtained from
   http://lessonresearch.net/NSF_TOOLKIT/Area_toolkit_manual.pdf
d) Mathematics Assessment Project (www.mathshell.org) A Formative Assessment Lesson: Evaluating Conditions for Congruency.

About the Unit and the Lesson

According to the Junior Certificate Mathematics Syllabus for examination from 2016 (Strand 2, Section 2.1 Synthetic geometry, pages 17-19) students at Ordinary Level:

a) learn about the concept of congruent triangles (see Syllabus: Geometry Course section 9.1, p. 80),
b) study Axiom 4: Congruent triangles (SAS, ASA and SSS),
c) complete constructions 10 (triangle, given lengths of three sides), 11 (triangle, given SAS data), 12 (triangle, given ASA data).

Furthermore, "it is intended that all of the geometrical results on the course would first be encountered by students through investigation" (Geometry Course, section 8.2).

From Junior Certificate Mathematics Guidelines for Teachers (2002, p.20): "Synthetic geometry is traditionally intended to promote students' ability to recognise and present logical arguments. More able students address one of the greatest of mathematical concepts, that of proof, and hopefully come to appreciate the abstractions and generalisations involved. Other students may not consider formal proof, but should be able to draw appropriate conclusions from given geometrical data. Explaining and defending their findings, in either case, should help students to further their powers of communication."

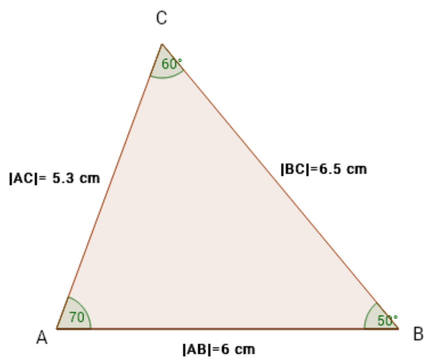
In the proposed sequence of lessons on the notion of congruence, at first students learn about congruent figures as those having the same shape and the same size (match up perfectly, manipulation of concrete materials, for example CDs, stack of A4 sheets, triangles made out of geostrips, pentagons). Then students identify the common features of congruent polygons (angles, side lengths). In the research lesson they minimise the amount of information required to draw congruent triangles and so arrive at the conditions SSS, SAS, ASA. They also practice their skills in constructing triangles. Further, through the construction of counterexamples, students draw conclusions about the AAA and SSA conditions.

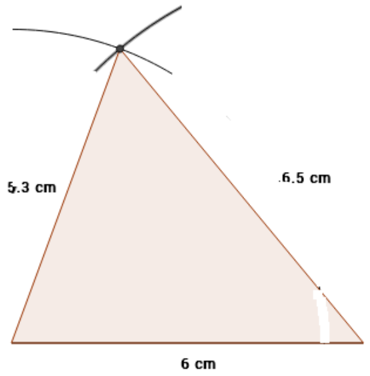
Flow of the Unit

Synthetic Geometry 2, First Year Teacher Handbook based on 2016 syllabus

Topic	# of lesson periods
Types of triangles and determining whether it is possible to draw a triangle with the given sets of conditions (including constructions 10, 11, 12, **without using SSS, SAS, ASA abbreviations,** and sets of conditions that do not yield a triangle).	3
The meaning of congruent shapes.	1
Congruent triangles (drawing congruent triangles and finding the minimal conditions SSS, ASA, SAS; realising that we cannot always draw congruent triangles using any three measurements; practice problems).	3 **(1 research lesson included)**
Theorem 2 (Isosceles triangles). Alternate angles. Theorems 3 (transversal) and 4 (sum of angles). Corresponding angles. Theorems 5 (transversal) and 6 (exterior angle in a triangle). Constructions 1-4	5
Translations, axial symmetry, central symmetry, rotation (map a triangle onto a congruent triangle)	5

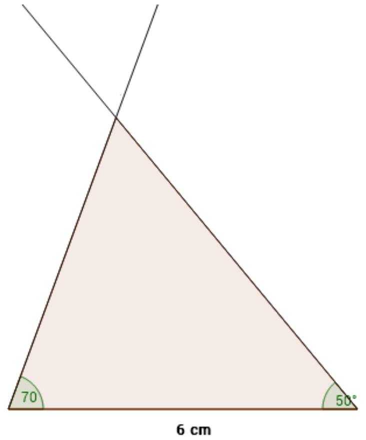
Flow of the Lesson

Teaching Activities and Students' Anticipated Responses	Points of Consideration
1. Introduction (3 minutes) What do we mean when we say that two geometric figures are congruent? (congruent figures are of the same shape and size) What is the same in these two triangles? (All three corresponding sides and all three corresponding angles are Problem for today: **What is the least amount of information we need to know to make sure that two triangles are congruent?** equal.) **2. Posing the Task (5 minutes)** On the worksheet you are given a triangle ABC. Your task is to draw a triangle that is congruent to triangle ABC.  You will all start by drawing a line segment that has the same length as the side AB. Then you will think about how many sides and angles you need to know in order to draw a triangle congruent to triangle ABC. You have ten minutes to work on this problem. Use your rulers, compasses and protractors.	Teacher shows pairs of geometric figures (rectangles, circles, pentagons, triangles) to allow students identify the common features during the comparison. Ensure in the preceding lessons that the students are comfortable with the word *corresponding.* Students use a compass, ruler & protractor.
3. Students individual work (10 minutes)	Teacher circulates the room assessing students' work to plan how to orchestrate the presentation of students' work on the board and class discussion.
4. Class discussion (15 minutes) Let's discuss now how you decided where the third vertex of the triangle should be placed. *Student 1*	Teacher picks a student who tried three sides. When the student has presented his/her reason, the teacher places a poster on the board.



If we know that **three sides** in one triangle are the same lengths as **three sides** in another triangle then the two triangles must be congruent.

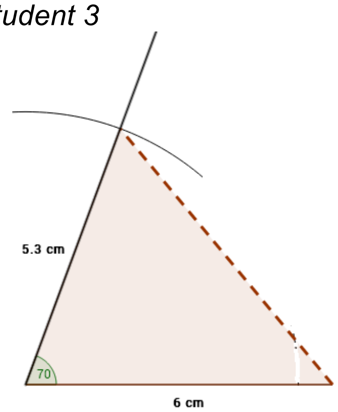
Student 2



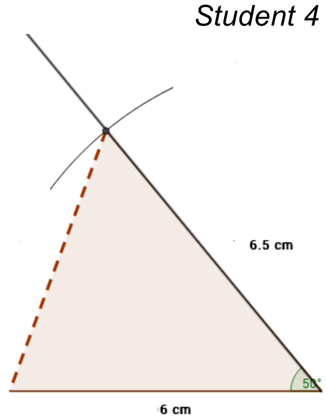
If **two angles and the included side** in one triangle are the same as **two angles and the included side** in another triangle, then the two triangles must be congruent.

Could we get away with one angle?

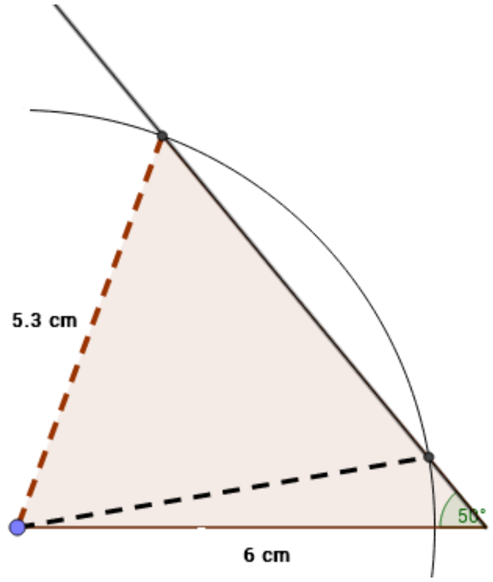
Student 3



Student 4



If **two sides and the angle between them** (included angle) in one triangle are equal to **two sides and the included angle** in another triangle then the two triangles must be congruent.

Let's think about what will happen if we take the 5.3 cm and 6 cm sides and the **non-included angle** of 50 degrees.  Could we have the SSA axiom? Given **two sides and a non-included angle**, how many different triangles can you make? (at least two) Could we get away without any side length? Let's take the three angles, 50, 60 and 70 degrees. How many different triangles can you make using these angles? (Everyone could have a different triangle.) Could we have the AAA axiom? (No)	If time does not allow leave it for the next class.
5. Summing up (2 minutes) Now I would like you to write down in your copies two sentences about what you learned in today's lesson.	

Evaluation: Plan for observing students

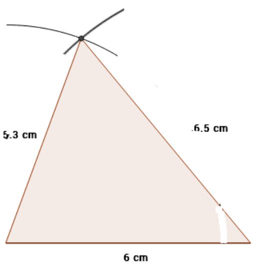
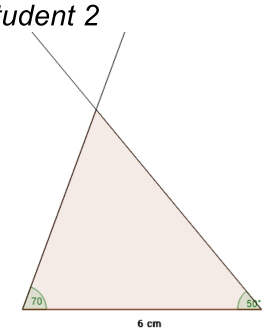
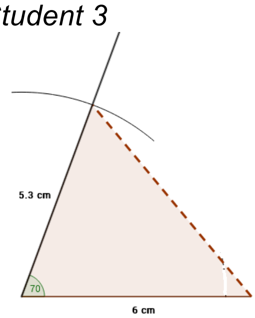
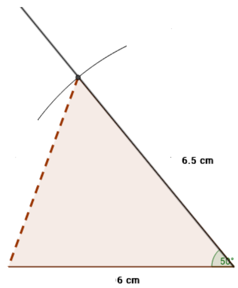
- A seating plan provided by the teacher
- Three observers, 8 students per observer, one observer uses the app LessonNote, the other two pen and paper using the Lesson Plan
- Types of student thinking and behaviour observers will focus on:

Introduction, posing the task	Can students recall the characteristics of congruent shapes? Was wording of the task clear? Questions asked by students
Individual work	Can students draw congruent triangles? Are prompts required? What strategies do they employ when drawing congruent triangles? Are they able to determine the minimum information needed? How long do students spend on the task?

	What kind of questions do students ask? Do they persist with the task?
Discussion	Are students attentive to what is happening on the board? Are clarifications needed to presenters' board work? Did the discussion promote student learning?

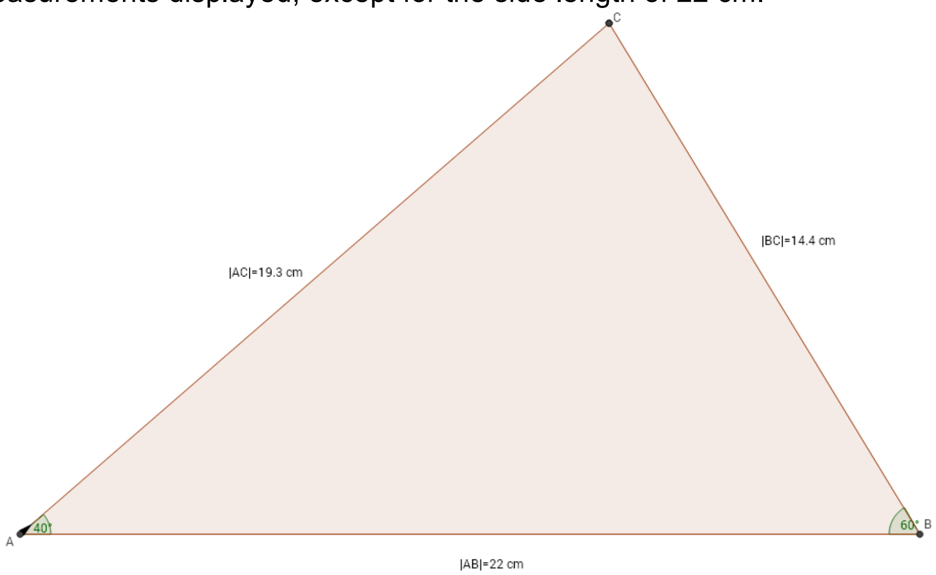
- Additional kinds of evidence collected (exemplars of students' work, photographs, end of lesson reflection)

Board Plan

SSS poster	**ASA poster**	**SAS poster**	
Student 1 	*Student 2* 	*Student 3* 	*Student 4* 

Post-lesson reflection

Note: The dimensions of a triangle given in class (see below) were different to the dimensions in the lesson plan. Furthermore, the diagram distributed among students had no measurements displayed, except for the side length of 22 cm.



What are the major patterns and tendencies in the evidence?

It is vital that students are aware of what they are trying to solve i.e. initial question must be very clear. Students must have clear understanding of what they are asked to do.

We found that when the students were presented with a line on the page they tended to come up with the ASA condition first. When they got this initial success and were prompted to look for other conditions appropriate to establish congruence they were more confident and clear about what they were asked to do. The two other conditions of SSS and SAS soon followed.

It was also observed that many students started their work by measuring the lengths and angles of the given triangle (measurements had not been given in the diagram).

What are the key observations or representative examples of student learning and thinking?

Students were slow to start and cautious at the outset but once they established one of the criteria to establish congruence they were more comfortable at attempting other conditions. At the start a student asked for clarification: “Do we count the bottom line?” as part of the minimal number of conditions.

One quarter of the class members were very clear from the outset of what was required. When the first student peer taught ASA to his classmates, reinforced by the teacher, all students in the class were then very clear on what was required and then they soon began to see two other sets of conditions as SAS and SSS.

When SSS was established as an acceptable criterion some students began to wonder if AAA would imply congruence. This misconception was very quickly ruled out by several students as not implying or proving congruency (“Angles are the same but sides could be anything”). One student asked “Is it possible to do a triangle using two bits of information?” That was quickly ruled out too by the rest of the class (given two side lengths “you wouldn’t know in which direction to draw the line”, “you wouldn’t know where the third side will connect”, given a side and an angle “you wouldn’t know how long the line will be”).

What does the evidence suggest about student thinking such as their misconceptions, difficulties, confusion, insights, surprising ideas, etc?

Students were clear about their understanding of congruent shapes. They understood what congruent triangles were.

The task of trying to come up with the least number of conditions required in order for two triangles to be congruent took some students a few minutes to comprehend. But, once they reached one condition, others then followed. Surprisingly, the majority of the students got the ASA condition first. A couple of students started off drawing the SSS condition first. It was noticed that one student took a while to start off but after positive reinforcement and prompting by the teacher the student then started to draw the condition ASA.

One student introduced AAA towards the end of class. This was then very successfully discussed and the students then ruled it out as a possible condition.

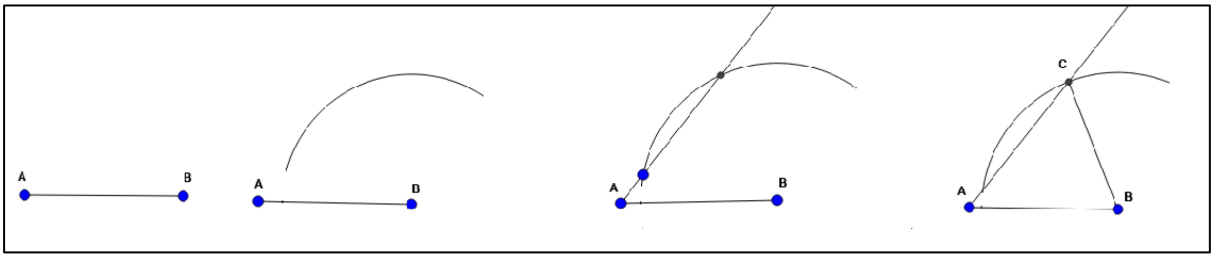
In what ways did students achieve or not achieve the learning goals?

Students achieved their learning goals as all students drew out at least one condition. Overall approximately half of the students in the class arrived at the three conditions ASA, SAS and SSS. The students were able to articulate their methods clearly at the board to their peers. The main concepts were reinforced and summarized by the teacher after each student presented their work. The pre-prepared posters worked very well in getting all students to follow the flow of the lesson. The final board plan worked very well as a final revision of the class. Overall, it was clear from the student engagement in this class that the students understood the concepts.

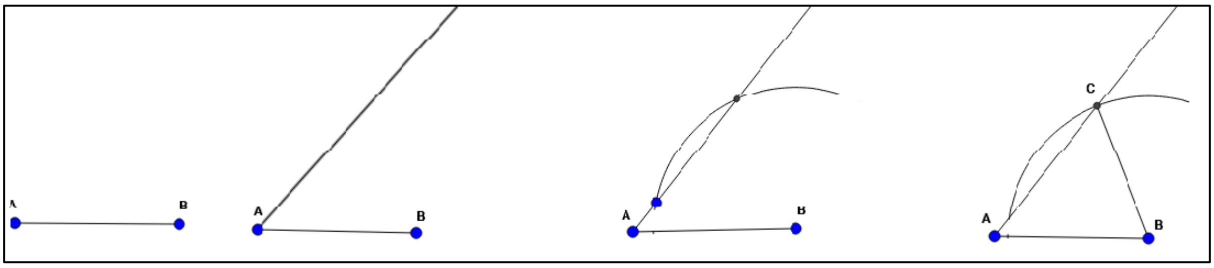
Based on your analysis, how would you change or revise the lesson?

A. Students showed that they are able to minimise the number of conditions needed to draw congruent triangles. There was evidence of students trying to draw a congruent triangle using SSA that would lead to two scenarios (Note: the dimensions given in class were different to the dimensions in the lesson plan, but led to the same observations). Five minutes into the task two students in a group of eight performed the following steps.

First student:



Second student:



Both students took the further point of intersection disregarding the other point. They both used two sides and a non-included angle. The time constraint of a 40 minute lesson did not allow for detailed discussion of the SSA condition.

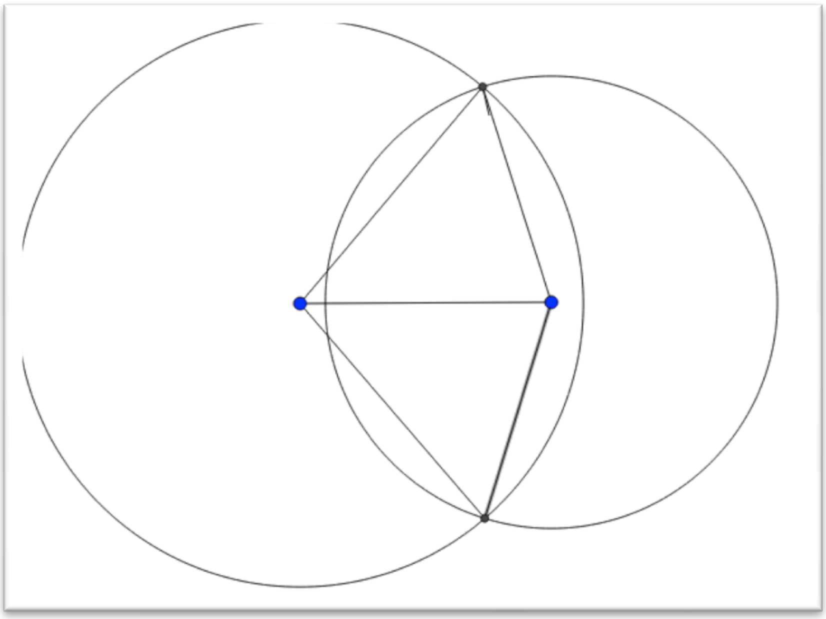
<u>Recommendation:</u> Return to the ambiguous case in the following lesson. Drawing longer arcs or even full circles may help to train student's eye to notice unusual cases.

B. Asking the students to justify their methods is important: "How could you convince your classmates that your triangle is congruent to triangle ABC?"

<u>Recommendation:</u> Let students think about ways to verify their observations. Students could check the lengths of the remaining sides and the size of the remaining angles or they could perform their constructions on tracing paper and then show that their triangles overlap with the triangle ABC.

C. Students tend to draw triangles that have similar orientation to the given triangle. Furthermore, some students do not see an arc as a part of a circle.

Recommendation: To make students aware of other choices, suggest drawing full circles, when using the three lengths (SSS). This approach would create two positions of the third vertex.



D. The lesson was closed with the whole class overview of the conditions established during the discussion.

   Recommendation: An alternative closure could involve the end of lesson reflection "Write down what you learnt today" followed up with a suitable homework (see Appendix 2).

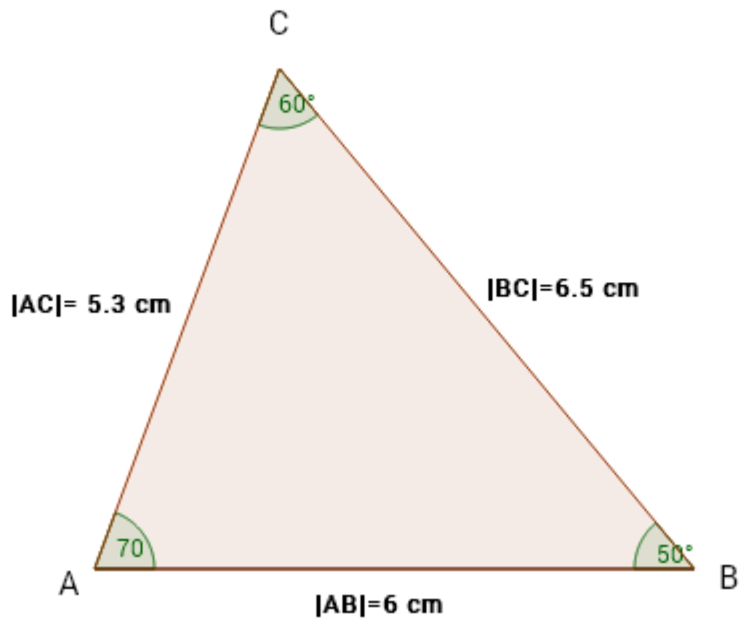
What are the implications for teaching in your field?

Through the process of planning the lesson we experienced how important it is (and not straightforward at all!) to formulate a clear meaningful problem that allows students the opportunity to extend their knowledge and give an account of their work within a single lesson. The amount of discussion and thinking about the shape of the lesson will certainly influence our future teaching. We learned that problem solving is not the same as just solving a task and active learning is not all about rolling a die and using unifix cubes. We understand that problem solving involves engaging in a task for which the solution method is not known beforehand, getting stuck and unstuck and communicating methods/solutions.

We sometimes lack conviction on students' ability to investigate a mathematical problem. Here, students in a mixed-ability class showed that they are able to draw congruent triangles by investigating the necessary side lengths and angle sizes. They showed a lot of enthusiasm at the end of the lesson formulating hypotheses regarding the possibility to further minimise the number of necessary conditions to two. It is likely that students would enjoy investigating a similar problem for congruent quadrilaterals.

It remains to be seen if the experience of this lesson will help retain information regarding congruent triangles and make dealing with abstract problems less daunting. However, we believe that this lesson deepened students understanding of the characteristics of congruent triangles and provided them with an opportunity to present **their own** methods (and thus gain the ownership of the SSS, SAS and ASA axioms). It would be worthwhile to prepare more lessons in a similar fashion.

Appendix 1. Worksheet

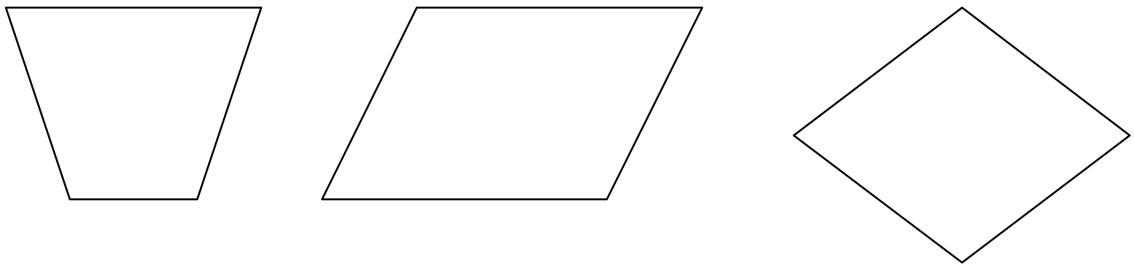


1. What is the least number of measurements that you need in order to draw a triangle that is congruent to triangle ABC?
2. Start by drawing a line segment that has the same length as side AB.

Appendix 2. Homework

1. (If alternate angles and basic properties of quadrilaterals studied before)

Which of the quadrilaterals below will make two congruent triangles when divided along the diagonal? Explain congruence without measuring the angles and sides.



Trapezium Parallelogram Rhombus

2. (Leading to counterexamples that could be discussed in the following lesson)

a) I have drawn a triangle. It has sides of length 3 cm and 4 cm and an angle of 34 degrees.

You draw a triangle with the same properties.
Must my triangle be congruent to yours?
Write down a convincing explanation.

b) I have drawn a triangle. It has angles of 50 and 30 degrees and one side is 5 cm long.

You draw a triangle with the same properties.
Must my triangle be congruent to yours?
Write down a convincing explanation.