WS07.02 Exponential Functions

Aim: To study the properties of exponential functions and learn the features of their graphs

Section A - Activity 1: The Exponential Function, $f(x) = 2^x$.

1. For $f(x) = 2^x$:

(i) The base of $f(x) = 2^x$ is ______________________

(ii) The exponent of $f(x) = 2^x$ is ______________________

(iii) What is varying in the function $f(x) = 2^x$? ______________________

(iv) What is constant in the function $f(x) = 2^x$? ______________________

2. For $f(x) = 2^x$:
What are the possible inputs i.e. values for x (the domain)?

Natural numbers ☐ Irrational numbers ☐
Integers ☐ Real numbers ☐
Rational numbers ☐

3. Set up a table of values and draw the graph of $f(x) = 2^x$ on your whiteboard:

X	2^x	$y = f(x)$
-4	2^{-4}	$\frac{1}{16}$
-3		
-2		
-1		
0		
1		
2		
3		
4		

4. Describe the graph of $f(x) = 2^x$:

(i) Is it a straight line? ______________________

(ii) Is $y = f(x)$ increasing or decreasing as x increases? ______________________

(iii) From the table above, find the average rate of change over different intervals.
For example from -2 to -1 and 2 to 3.

What do you notice? ______________________

____

(iv) Describe how the curvature/rate of change is changing. ______________________

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5. For $f(x) = 2^x$:

(i) What are the possible outputs (range) for $f(x) = 2^x$. ____________________

(ii) Is it possible to have negative outputs? Explain why? ____________________

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(iii) What happens to the output as x decreases? ____________________

____

(iv) Is an output of 0 possible? Why do you think this is? ____________________

____

(v) What are the implications of this for the x-intercept of the graph? ____________

____

(vi) What is the y-intercept of the graph of $f(x) = 2^x$? ____________________

____

Section A – Activity 2: The Exponential Function, $g(x) = 3^x$.

1. For $g(x) = 3^x$:

(i) The base of $g(x) = 3^x$ is ______________________

(ii) The exponent of $g(x) = 3^x$ is ______________________

(iii) What is varying in the function $g(x) = 3^x$? ______________________

(iv) What is constant in the function $g(x) = 3^x$? ______________________

2. For $g(x) = 3^x$:
What are the possible inputs i.e. values for x (the domain)?

Natural numbers ☐ Irrational numbers ☐
Integers ☐ Real numbers ☐
Rational numbers ☐

3. Set up a table of values and draw the graph of $g(x) = 3^x$ on your whiteboard:

X	3^x	$y = g(x)$
-4	3^{-4}	$1/81$
-3		
-2		
-1		
0		
1		
2		
3		
4		

4. In relation to the graph of $g(x) = 3^x$:

(i) Is it a straight line? ______________________

(ii) Is $y = g(x)$ increasing or decreasing as x increases? ______________________

(iii) From the table above, find the average rate of change over different intervals.
For example from -2 to -1 and 2 to 3.

What do you notice? ______________________

____

(iv) Describe how the curvature/rate of change is changing. ______________________

____

5. For $g(x) = 3^x$:

(i) What are the possible outputs (range) for $g(x) = 3^x$. ____________________

(ii) Is it possible to have negative outputs? Explain why? ____________________

(iii) What happens to the output as x decreases? ____________________

(iv) Is an output of 0 possible? Why do you think this is? ____________________

(v) What are the implications of this for the x-intercept of the graph? ____________________

(vi) What is the y-intercept of the graph of $g(x) = 3^x$? ____________________

Section A – Activity 3: Compare the graph of $f(x) = 2^x$ *with the graph of* $g(x) = 3^x$.

1. How are they similar and how do they differ? ______________________

____

____

2. Consider the relations $\{(x, y) | x \in \mathbb{R},\ y \in \mathbb{R},\ y = 2^x\}$ and $\{(x, y) | x \in \mathbb{R},\ y \in \mathbb{R},\ y = 3^x\}$.

   Are they functions? Explain. ______________________

____

____

3. What name do you think is given to this type of function and why do you think it is given this name?

____

____

Section A – Activity 4: Understand the characteristics of $f(x) = a^x,\ a > 1$.

1. What is the domain of $f(x) = a^x,\ a > 1$? ______________________

2. In relation to the graph of $f(x) = a^x,\ a > 1$.

   (i) Is it a straight line? ______________________

   (ii) Is $y = f(x)$ increasing or decreasing as x increases? ______________________

   (iii) Does it have a maximum value? ______________________

   (iv) Does it have a minimum value? ______________________

   (v) Describe how its curvature/rate of change is changing. ______________________

____

3. What is the range of $f(x) = a^x,\ a > 1$? ______________________

4. What is the x-intercept of the graph $f(x) = a^x,\ a > 1$? ______________________

5. What is the y-intercept of the graph $f(x) = a^x,\ a > 1$? ______________________

Section B – Activity 1: The Exponential Function, $f(x)=\left(\frac{1}{2}\right)^x$.

1. For $f(x)=\left(\frac{1}{2}\right)^x$:

(i) The base of $f(x)=\left(\frac{1}{2}\right)^x$ is ____________________

(ii) The exponent of $f(x)=\left(\frac{1}{2}\right)^x$ is ____________________

(iii) What is varying in the function $f(x)=\left(\frac{1}{2}\right)^x$? ____________________

(iv) What is constant in the function $f(x)=\left(\frac{1}{2}\right)^x$? ____________________

2. For $f(x)=\left(\frac{1}{2}\right)^x$:

What are the possible inputs i.e. values for x (the domain)?

Natural numbers	☐	Irrational numbers	☐
Integers	☐	Real numbers	☐
Rational numbers	☐		

3.

Set up a table of values and draw the graph of $f(x)=\left(\frac{1}{2}\right)^x$ on your whiteboard:

X	$\left(\frac{1}{2}\right)^x$	$y=f(x)$
-4	$\left(\frac{1}{2}\right)^{-4}$	16
-3		
-2		
-1		
0		
1		
2		
3		
4		

4. In relation to the graph of $f(x) = \left(\frac{1}{2}\right)^x$:

(i) Is it a straight line? ____________________

(ii) Is $y = f(x)$ increasing or decreasing as x increases? ____________________

(iii) From the table above, find the average rate of change over different intervals. For example from -2 to -1 and 2 to 3.

What do you notice? ____________________

____

(iv) Describe how the curvature/rate of change is changing. ____________________

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5. For $f(x) = \left(\frac{1}{2}\right)^x$:

(i) What are the possible outputs (range) for $f(x) = \left(\frac{1}{2}\right)^x$. ____________________

(ii) Is it possible to have negative outputs? Explain why? ____________________

____

(iii) What happens to the output as x decreases? ____________________

____

(iv) Is an output of 0 possible? Why do you think this is? ____________________

____

(v) What are the implications of this for the x-intercept of the graph? ____________________

____

(vi) What is the y-intercept of the graph of $f(x) = \left(\frac{1}{2}\right)^x$? ____________________

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Section B – Activity 2: The Exponential Function, $g(x)=\left(\frac{1}{3}\right)^x$.

1. For $g(x)=\left(\frac{1}{3}\right)^x$:

(i) The base of $g(x)=\left(\frac{1}{3}\right)^x$ is ______________________________

(ii) The exponent of $g(x)=\left(\frac{1}{3}\right)^x$ is ______________________________

(iii) What is varying in the function $g(x)=\left(\frac{1}{3}\right)^x$? ______________________________

(iv) What is constant in the function $g(x)=\left(\frac{1}{3}\right)^x$? ______________________________

2. For $g(x)=\left(\frac{1}{3}\right)^x$:

What are the possible inputs i.e. values for x (the domain)?

- Natural numbers ☐
- Integers ☐
- Rational numbers ☐
- Irrational numbers ☐
- Real numbers ☐

3.

Set up a table of values and draw the graph of $g(x)=\left(\frac{1}{3}\right)^x$ below on your whiteboard:

X	$\left(\frac{1}{3}\right)^x$	$y=g(x)$
–4	$\left(\frac{1}{3}\right)^{-4}$	81
–3		
–2		
–1		
0		
1		
2		
3		
4		

4. In relation to the graph of $g(x)=\left(\frac{1}{3}\right)^x$:

(i) Is it a straight line? ______________________________

(ii) Is y increasing or decreasing as x increases? ______________________________

(iii) From the table above, find the average rate of change over different intervals. For example from -2 to -1 and 2 to 3.

What do you notice? ______________________________

____

(iv) Describe how the curvature/rate of change is changing. ______________________________

____

5. For $g(x)=\left(\frac{1}{3}\right)^x$:

(i) What are the possible outputs (range) for $g(x)=\left(\frac{1}{3}\right)^x$. ______________________________

(ii) Is it possible to have negative outputs? Explain why? ______________________________

____

(iii) What happens to the output as x decreases? ______________________________

____

(iv) Is an output of 0 possible? Why do you think this is? ______________________________

____

(v) What are the implications of this for the x-intercept of the graph? ______________________________

____

(vi) What is the y-intercept of the graph of $g(x)=\left(\frac{1}{3}\right)^x$? ______________________________

____

Section B - Activity 3: Compare the graph of $f(x)=\left(\frac{1}{2}\right)^x$ with the graph of $g(x)=\left(\frac{1}{3}\right)^x$.

1. How are they the same and how do they differ? ______________________________

____

____

2. Consider the relations $\left\{(x,y)\middle|\, x\in\mathbb{R},\ y\in\mathbb{R},\ y=\left(\frac{1}{2}\right)^x\right\}$ and $\left\{(x,y)\middle|\, x\in\mathbb{R},\ y\in\mathbb{R},\ y=\left(\frac{1}{3}\right)^x\right\}$.

Are they functions? Explain. ______________________________

____

____

3. What name do you think we give to this type of function and why do you think it is given this name?

____

____

Section B - Activity 4: Understand the characteristics of $f(x)=a^x$, $0<a<1$.

1. What is the domain of $f(x)=a^x$, $0<a<1$? ______________________________

2. In relation to the graph of $f(x)=a^x$, $0<a<1$.

(i) Is it a straight line? ______________________________

(ii) Is $y=f(x)$ increasing or decreasing as x increases? ______________________________

(iii) Does it have a maximum value? ______________________________

(iv) Does it have a minimum value? ______________________________

(iv) Describe how its curvature/rate of change is changing. ______________________________

____

3. What is the range of $f(x)=a^x$, $0<a<1$? ______________________________

4. What is the x-intercept of the graph $f(x)=a^x$, $0<a<1$? ______________________________

5. What is the y-intercept of the graph $f(x)=a^x$, $0<a<1$? ______________________________

Note: For all the following, you should assume that the domain is $\mathbb{R}$.

Section C - Activity 1: Compare the graph of $f(x) = 2^x$ with the graph of $f(x) = \left(\frac{1}{2}\right)^x$.

1. How are the graphs similar? ____

____

2. How are the graphs different? ____

____

3. Rewrite $f(x) = \left(\frac{1}{2}\right)^x$ in the form $f(x) = 2^k$. ____________________________

4. What transformation maps the graph of $f(x) = 2^x$ onto the graph of $f(x) = \left(\frac{1}{2}\right)^x$?

____

Section C - Activity 2: Compare the graph of $g(x) = 3^x$ with the graph of $g(x) = \left(\frac{1}{3}\right)^x$.

1. How are the graphs similar? ____

____

2. How do the graphs differ? ____

____

3. Rewrite $g(x) = \left(\frac{1}{3}\right)^x$ in the form $g(x) = 3^k$. ____________________________

4. What transformation maps the graph of $g(x) = 3^x$ onto the graph of $g(x) = \left(\frac{1}{3}\right)^x$?

____

1. If $f(x) = a^x$, $a \in \mathbb{R}$, $a > 1$, then the ***properties*** of the exponential ***function*** are:

2. If $f(x) = a^x$, $a \in \mathbb{R}$, $a > 1$, then the ***features*** of the exponential ***graph*** are:

3. If $f(x) = a^x$, $a \in \mathbb{R}$, $0 < a < 1$, then the ***properties*** of the exponential ***function*** are:

4. If $f(x) = a^x$, $a \in \mathbb{R}$, $0 < a < 1$, then the ***features*** of the exponential ***graph*** are:

Section C – Activity 4: Which of the following equations represent exponential functions?

Function	Is it an exponential Function? Yes/No	Reason
$f(x) = \left(\frac{1}{2}\right)^x$		
$f(x) = x^2$		
$f(x) = (-2)^x$		
$f(x) = 2(3)^x$		
$f(x) = -2^x$		
$f(x) = 3(x)^{\frac{1}{2}}$		
$f(x) = (0.9)^x$		

Note: Extension Activities are required to strengthen students' abilities in the following areas from the syllabus:

Level	*Syllabus*	*Page*
JCHL	$f(x) = a2^x$ and $f(x) = a3^x$, where $a \in \mathbb{N}, x \in \mathbb{R}$.	Page 31
LCFL	$f(x) = a2^x$ and $f(x) = a3^x$, where $a \in \mathbb{N}, x \in \mathbb{R}$.	Page 32
LCOL	$f(x) = ab^x$, where $a \in \mathbb{N}$, $b, x \in \mathbb{R}$.	Page 32
LCHL	$f(x) = ab^x$, where a, b, $x \in \mathbb{R}$.	Page 32

1. A cell divides itself into two every day. The number of cells C after D days is obtained from the function:

$$C = 2^D$$

(a) Draw a graph of the function for $0 \le D \le 6$.
(b) Find the number of cells after 15 days.

2. The value of a mobile phone M (in cents) after T years can be obtained from the following function:

$$M = k\left(\frac{1}{2}\right)^T, \text{where } k \text{ is a constant.}$$

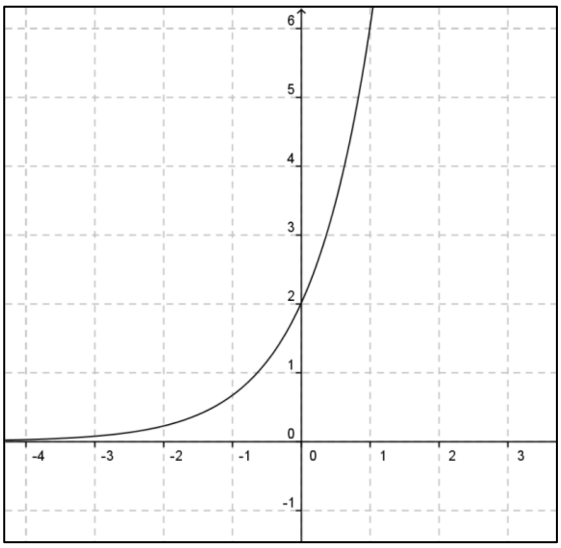
(a) Draw a graph of the function for $0 \le T \le 6$.
(b) Find the value of k given that the value of the mobile phone after 3 years is €100.
(c) Find the value of the phone after 7 years.

3. The number of bacteria B in a sample after starting an experiment for m minutes is given by:

$$B = 50(3)^{0.04m}$$

(a) Find the number of bacteria in the sample at the start of the experiment.
(b) Find the number of bacteria in the sample after starting the experiment for 3 hours.

4. The graph of $f(x) = ka^x$ is shown:
(a) Find the value of k and a.
(b) Hence find the value of $f(x)$ when $x = 8$.



Q5. Olive finds that the number of bacteria in a sample doubles every 5 hours. Originally there are 8 bacteria in the sample.

Complete the table below:

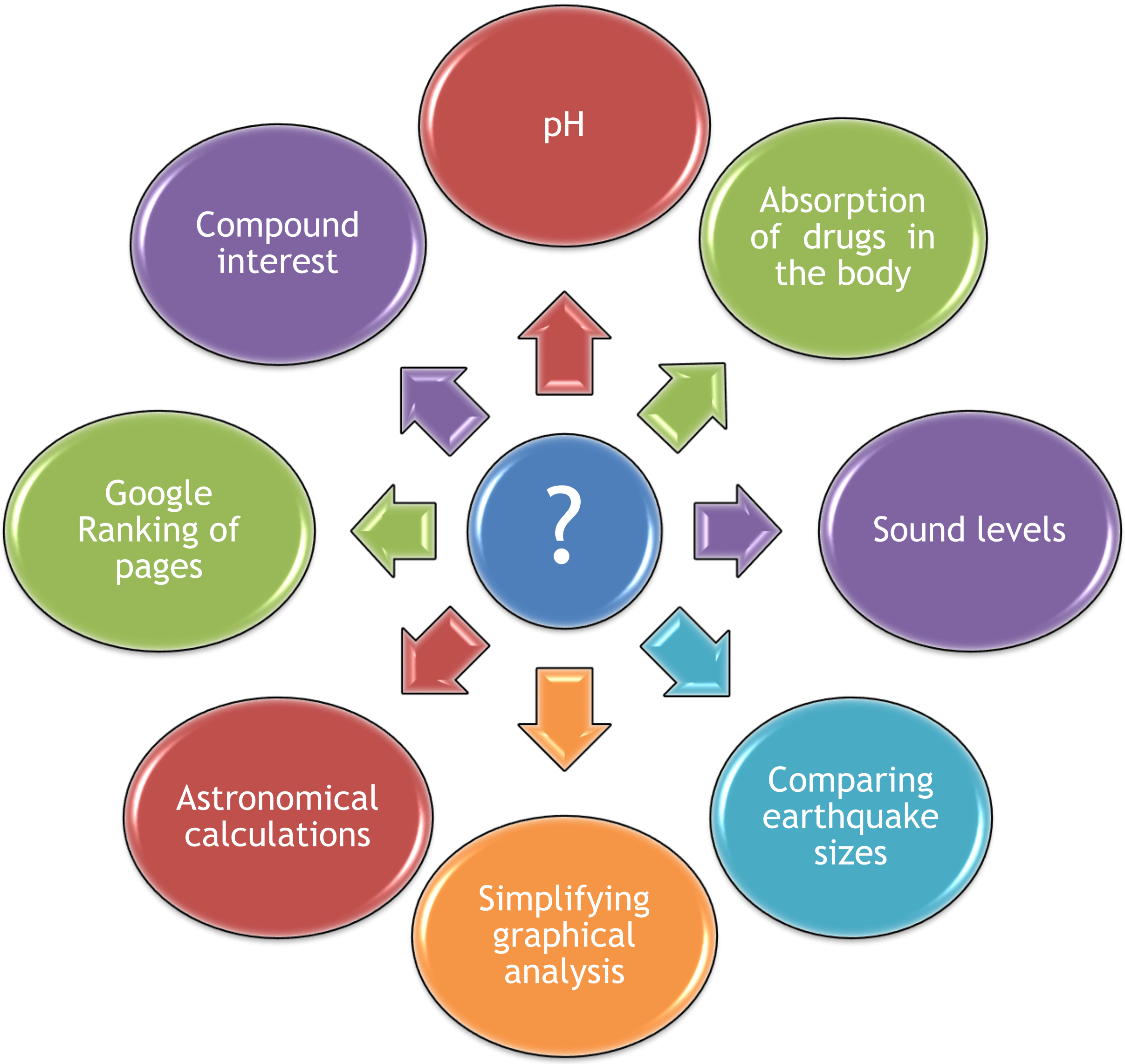
Number of hours (*hrs.*)	Number of bacteria (*b*)
0	8
5	
10	
15	

(a) Express *b* in terms of *h*.
(b) Find the number of bacteria in the sample after 13 hours.
(c) How many hours later will the number of bacteria be more than 100.

Q6. When a microwave oven is turned on for x minutes the relationship between the temperature C° inside the oven is given by $C(x) = 500 - 480(0.9)^x$ where $x \geq 0$.
(a) Find the value of $C(0)$.
(b) Explain the meaning of $C(0)$.
(c) Can the temperature inside the microwave oven reach $550C^\circ$?

Answers

Q1 (b) 32,768 cells, **Q2 (b)** €800 **(c)** €6.25, **Q3 (a)** 50 **(b)** 136,220 bacteria, **Q4 (a)** $k = 2$, $a = 3$ **(b)** 13,122, **Q5 (b)** $b = 8(2)^{\frac{h}{5}}$ **(c)** 48 bacteria **(d)** 18.22 hrs., **Q6 (a)** 20° C **(c)** No


pH
Compound interest
Absorption of drugs in the body
?
Google Ranking of pages
Sound levels
Astronomical calculations
Comparing earthquake sizes
Simplifying graphical analysis

Activity 1 Making the Most of a Euro

Invest €1 for 1 year at 100% compound Interest.

Investigate the change in the final value, if the annual interest rate of 100% is compounded over smaller and smaller time intervals.

The interest rate *i* per compounding period is calculated by dividing the annual rate of 100% by the number of compounding periods per year.

Compounding period	Final value, $F = P(1+i)^t$, where *i* is the interest rate for a given compounding period and *t* is the number of compounding periods per year. Calculate *F* correct to 8 decimal places.
Yearly $i = 1$	$F = 1(1+1)^1 = 2$
Every 6 mths. $i = \frac{1}{2}$	$F = 1\left(1+\frac{1}{2}\right)^2 = 2.25$
Every 3 mths. $i =$	
Every mth. $i =$	
Every week. $i =$	
Every day. $i =$	
Every hour. $i =$	
Every minute. $i =$	
Every second. $i =$	

What if the compounding period was 1 millisecond (10^{-3} s), 1 microsecond (10^{-6} s) or 1 nanosecond (10^{-9} s)? What difference would it make?

Will *F* ever reach 3? How about 2.8?

Activity 2 Further Exploration of Exponential Functions

1. How long will it take for a sum of money to double if invested at 20% compound interest rate compounded annually?

2. 500 mg of a medicine enters a patient's blood stream at noon and decays exponentially at a rate of 15% per hour.

(i) Write an equation to express the amount remaining in the patient's blood stream at after t hours.

(ii) Find the time when only 25 mg of the original amount of medicine remains active.

3.

x	2^x	*y*
0	2^0	1
1	2^1	2
2	2^2	4
3	2^3	8
4	2^4	16
5	2^5	32
6	2^6	64
7	2^7	128
8	2^8	256
9	2^9	512
10	2^{10}	1024
11	2^{11}	2048
12	2^{12}	4096

(a) Describe the type of sequence formed by the numbers in the first column.

(b) Describe the type of sequence formed by the numbers in the second and third columns.

(c) Using the table, and your knowledge of indices, carry out the following operations of multiplication and division in the second sequence, linking the answer to numbers in the first sequence.

(i) 32×128

(ii) $4096 \div 512$

(iii) 8^4

x	2^x	y
13	2^{13}	8,192
14	2^{14}	16,384
15	2^{15}	32,768
16	2^{16}	65,536
17	2^{17}	13,1072
18	2^{18}	262,144
19	2^{19}	524,288
20	2^{20}	1,048,576
21	2^{21}	2,097,152
22	2^{22}	419,4304

x	2^x	y
23	2^{23}	838,8608
24	2^{24}	16,777,216
25	2^{25}	33,554,432
26	2^{26}	67,108,864
27	2^{27}	134,217,728
28	2^{28}	268,435,456
29	2^{29}	536,870,912
30	2^{30}	1,073,741,824
31	2^{31}	2,147,483,648
32	2^{32}	4,294,967,296

Using Different Bases

x	3^x
0	1
1	3
2	9
3	27
4	81
5	243
6	729
7	2,187
8	6,561
9	19,683
10	59,049

x	4^x
0	1
1	4
2	16
3	64
4	256
5	1,024
6	4,096
7	16,384
8	65,536
9	262,144
10	1,048,576

x	5^x
0	1
1	5
2	25
3	125
4	625
5	3,125
6	15,625
7	78,125
8	390,625
9	1,953,125
10	9,765,625

x	6^x
0	1
1	6
2	36
3	216
4	1,296
5	7,776
6	46,656
7	279,936
8	1,679,616
9	10,077,696
10	60,466,176

x	10^x
0	1
1	10
2	100
3	1,000
4	10,000
5	100,000
6	1,000,000
7	10,000,000
8	100,000,000
9	1,000,000,000
10	10,000,000,000

x	$\log_3(3^x)$
1	0
3	1
9	2
27	3
81	4
243	5
729	6
2,187	7
6,561	8
19,683	9
59,049	10

x	$\log_4(4^x)$
1	0
4	1
16	2
64	3
256	4
1,024	5
4,096	6
16,384	7
65,536	8
262,144	9
1,048,576	10

x	$\log_5(5^x)$
1	0
5	1
25	2
125	3
625	4
3,125	5
15,625	6
78,125	7
390,625	8
1,953,125	9
9,765,625	10

x	$\log_6(6^x)$
1	0
6	1
36	2
216	3
1,296	4
7,776	5
46,656	6
279,936	7
1,679,616	8
10,077,696	9
60,466,176	10

x	$\log_{10}(10^x)$
1	0
10	1
100	2
1,000	3
10,000	4
100,000	5
1,000,000	6
10,000,000	7
100,000,000	8
1,000,000,000	9
10,000,000,000	10

Formula and Tables Page 21

Séana agus logartaim — Indices and logarithms

$$a^p a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$\left(a^p\right)^q = a^{pq}$$

$$a^0 = 1$$

$$a^{-p} = \frac{1}{a^p}$$

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

$$(ab)^p = a^p b^p$$

$$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a\left(x^q\right) = q\log_a x$$

$$\log_a 1 = 0$$

$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

$$a^x = y \quad \Leftrightarrow \quad \log_a y = x$$

$$\log_a(a^x) = x$$

$$a^{\log_a x} = x$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$