Activity 2 Further Exploration of Exponential Functions

1. How long will it take for a sum of money to double if invested at 20% compound interest rate compounded annually?

2. 500 mg of a medicine enters a patient's blood stream at noon and decays exponentially at a rate of 15% per hour.
   (i) Write an equation to express the amount remaining in the patient's blood stream at after t hours.
   (ii) Find the time when only 25 mg of the original amount of medicine remains active.

3.

x	2^x	y
0	2^0	1
1	2^1	2
2	2^2	4
3	2^3	8
4	2^4	16
5	2^5	32
6	2^6	64
7	2^7	128
8	2^8	256
9	2^9	512
10	2^{10}	1024
11	2^{11}	2048
12	2^{12}	4096

(a) Describe the type of sequence formed by the numbers in the first column.

(b) Describe the type of sequence formed by the numbers in the second and third columns.

(c) Using the table, and your knowledge of indices, carry out the following operations of multiplication and division in the second sequence, linking the answer to numbers in the first sequence.
   (i) 32×128

   (ii) $4096 \div 512$

   (iii) 8^4

x	2^x	y	x	2^x	y
13	2^{13}	8,192	23	2^{23}	838,8608
14	2^{14}	16,384	24	2^{24}	16,777,216
15	2^{15}	32,768	25	2^{25}	33,554,432
16	2^{16}	65,536	26	2^{26}	67,108,864
17	2^{17}	13,1072	27	2^{27}	134,217,728
18	2^{18}	262,144	28	2^{28}	268,435,456
19	2^{19}	524,288	29	2^{29}	536,870,912
20	2^{20}	1,048,576	30	2^{30}	1,073,741,824
21	2^{21}	2,097,152	31	2^{31}	2,147,483,648
22	2^{22}	419,4304	32	2^{32}	4,294,967,296

Using Different Bases

x	3^x
0	1
1	3
2	9
3	27
4	81
5	243
6	729
7	2,187
8	6,561
9	19,683
10	59,049

x	4^x
0	1
1	4
2	16
3	64
4	256
5	1,024
6	4,096
7	16,384
8	65,536
9	262,144
10	1,048,576

x	5^x
0	1
1	5
2	25
3	125
4	625
5	3,125
6	15,625
7	78,125
8	390,625
9	1,953,125
10	9,765,625

x	6^x
0	1
1	6
2	36
3	216
4	1,296
5	7,776
6	46,656
7	279,936
8	1,679,616
9	10,077,696
10	60,466,176

x	10^x
0	1
1	10
2	100
3	1,000
4	10,000
5	100,000
6	1,000,000
7	10,000,000
8	100,000,000
9	1,000,000,000
10	10,000,000,000

x	$\log_3(3^x)$
1	0
3	1
9	2
27	3
81	4
243	5
729	6
2,187	7
6,561	8
19,683	9
59,049	10

x	$\log_4(4^x)$
1	0
4	1
16	2
64	3
256	4
1,024	5
4,096	6
16,384	7
65,536	8
262,144	9
1,048,576	10

x	$\log_5(5^x)$
1	0
5	1
25	2
125	3
625	4
3,125	5
15,625	6
78,125	7
390,625	8
1,953,125	9
9,765,625	10

x	$\log_6(6^x)$
1	0
6	1
36	2
216	3
1,296	4
7,776	5
46,656	6
279,936	7
1,679,616	8
10,077,696	9
60,466,176	10

x	$\log_{10}(10^x)$
1	0
10	1
100	2
1,000	3
10,000	4
100,000	5
1,000,000	6
10,000,000	7
100,000,000	8
1,000,000,000	9
10,000,000,000	10

Formula and Tables Page 21

Séana agus logartaim   Indices and logarithms

$$a^p a^q = a^{p+q}$$

$$\frac{a^p}{a^q} = a^{p-q}$$

$$\left(a^p\right)^q = a^{pq}$$

$$a^0 = 1$$

$$a^{-p} = \frac{1}{a^p}$$

$$a^{\frac{1}{q}} = \sqrt[q]{a}$$

$$a^{\frac{p}{q}} = \sqrt[q]{a^p} = \left(\sqrt[q]{a}\right)^p$$

$$\left(ab\right)^p = a^p b^p$$

$$\left(\frac{a}{b}\right)^p = \frac{a^p}{b^p}$$

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a\left(x^q\right) = q\log_a x$$

$$\log_a 1 = 0$$

$$\log_a\left(\frac{1}{x}\right) = -\log_a x$$

$$a^x = y \iff \log_a y = x$$

$$\log_a(a^x) = x$$

$$a^{\log_a x} = x$$

$$\log_b x = \frac{\log_a x}{\log_a b}$$

*Switching between Exponential and logarithmic forms of Equations*

4.

Evaluate the expression below forming an equation	Write the equivalent exponential form of the equation formed from the first column
$\log_2 16 = 4$	
$\log_2\left(\dfrac{1}{64}\right)$	
$\log_2(1)$	
$\log_2\left(\dfrac{1}{8}\right)$	
$\log_e e$	
$\log_2(-4)$	

5.

Exponential form of an equation	Write the equivalent log form of the equation in the previous column
$5^2 = 25$	
$5^{-2} = \dfrac{1}{25}$	
$10^1 = 10$	
$9^{\frac{1}{2}} = 3$	
$27^{\frac{1}{3}} = 3$	
$b^0 = 1$	

6. Cut out the equilateral triangles and using the rules for logs, line up sides having equivalent expressions to make a hexagon.
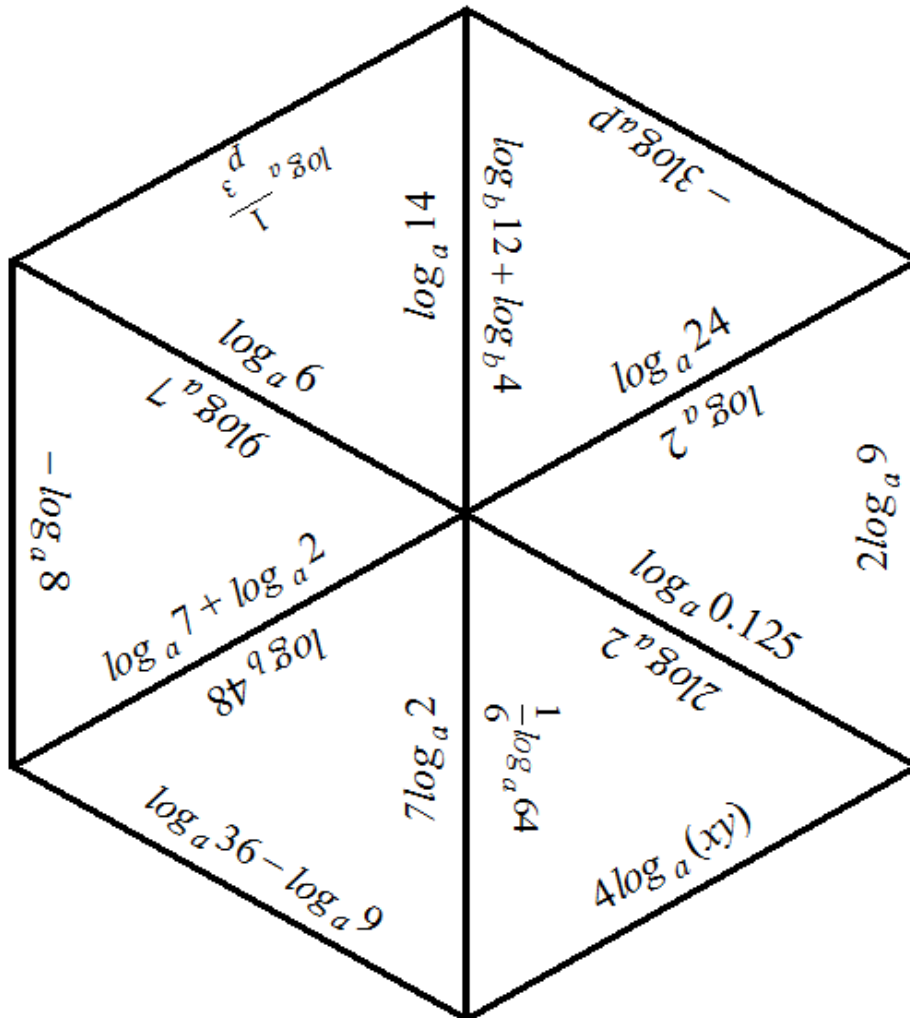
Note: This jigsaw was created using *Tarsia*, a free software package for creating numerous types of jigsaw and matching exercises.

7A. Evaluate each of the following:

$\log_2(32 \times 2) = \log_2(64) = $ ____ $\quad\big|\quad$ $\log_2(32) + \log_2(2) = $ ___ $+$ ___ $= $ ___

$\log_3(27 \times 9) = \log_3(243) = $ ____ $\quad\big|\quad$ $\log_3(27) + \log_3(9) = $ ___ $+$ ___ $= $ ___

$\log_5(25 \times 5) = \log_5(125) = $ ____ $\quad\big|\quad$ $\log_5(25) + \log_5(5) = $ ___ $+$ ___ $= $ ___

$\log_2\left(16 \times \dfrac{1}{16}\right) = \log_2(64) = $ ____ $\quad\big|\quad$ $\log_2(16) + \log_2\left(\dfrac{1}{16}\right) = $ ___ $+$ ___ $= $ ___

What pattern seems to hold?

Can you write a rule for $\log_b(xy)$ in terms of $\log_b(x)$ and $\log_b(y)$?

7B. Evaluate each of the following:

$\log_2(64 \div 4) = \log_2(16) = $ ____ $\quad\big|\quad$ $\log_2(64) - \log_2(4) = $ ___ $-$ ___ $= $ ___

$\log_6(216 \div 6) = \log_6(36) = $ ____ $\quad\big|\quad$ $\log_6(216) - \log_6(6) = $ ___ $-$ ___ $= $ ___

$\log_{10}(100 \div 1000) = \log_{10}\left(\dfrac{1}{10}\right) = $ ____ $\quad\big|\quad$ $\log_{10}(100) - \log_{10}(1000) = $ ___ $-$ ___ $= $ ___

$\log_5(25 \div 25) = \log_5(1) = $ ____ $\quad\big|\quad$ $\log_5(25) - \log_5(25) = $ ___ $-$ ___ $= $ ___

What pattern seems to hold?

Can you write a rule for $\log_b\left(\dfrac{x}{y}\right)$ in terms of $\log_b(x)$ and $\log_b(y)$?

7C. Evaluate each of the following:

$\log_2(8)^3 = \log_2(512) = $ ____ $\quad\big|\quad$ $3\log_2(8) = $ ___ $= 3(\ \ ) = $ ___

$\log_2(256)^{\frac{1}{2}} = \log_2(16) = $ ____ $\quad\big|\quad$ $\dfrac{1}{2}\log_2(256) = $ ___ $= \dfrac{1}{2}(\ \ ) = $ ___

$\log_{10}(10)^4 = \log_{10}(10,000) = $ ____ $\quad\big|\quad$ $4\log_{10}(10) = $ ___ $= 4(\ \ ) = $ ___

$\log_3(27)^2 = \log_3(729) = $ ____ $\quad\big|\quad$ $2\log_3(27) = $ ___ $= 2(\ \ ) = $ ___

What pattern seems to hold?

Can you write a rule for $\log_b(x)^y$ in terms of $\log_b(x)$?
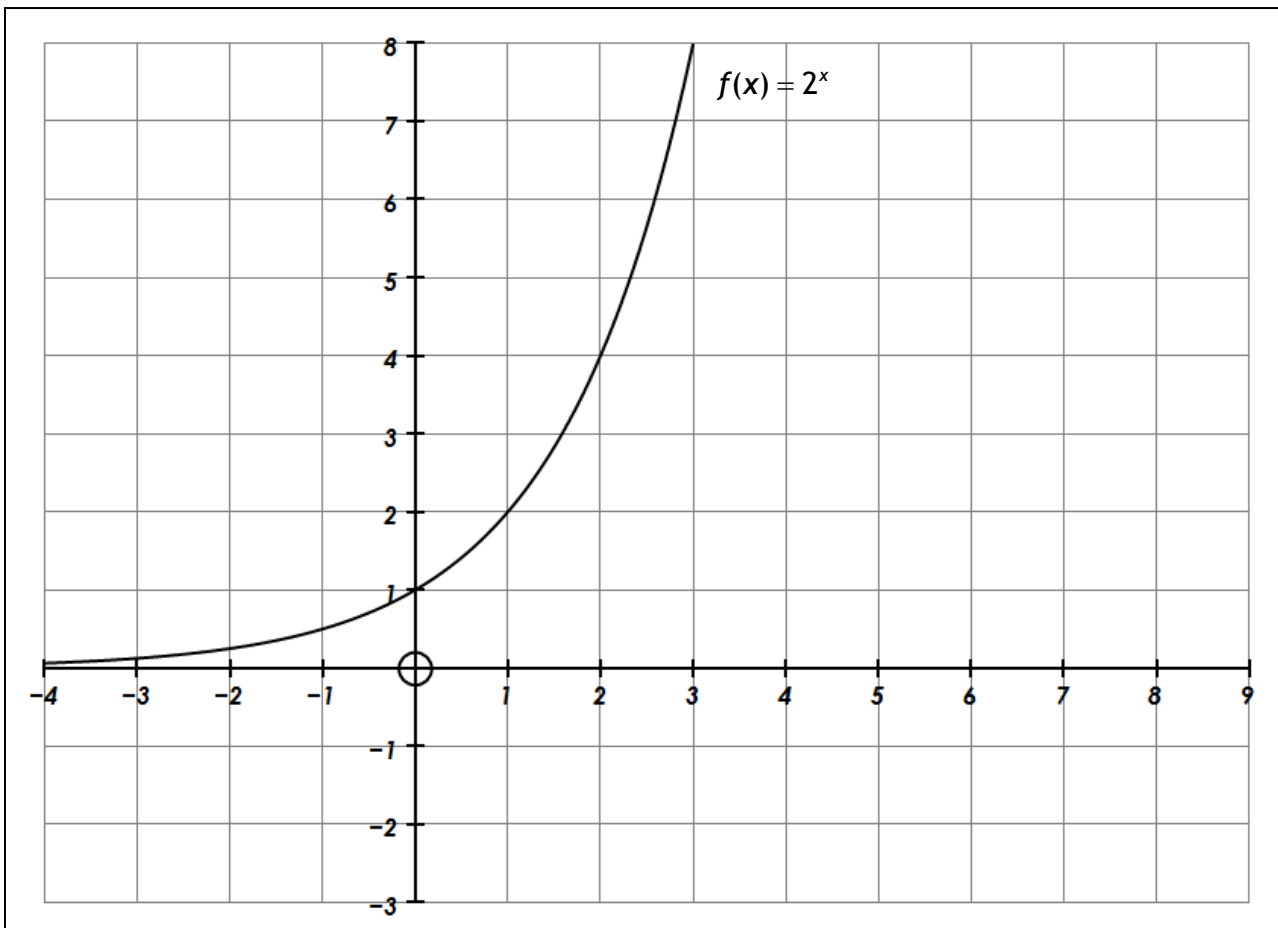
8. From discrete to continuous
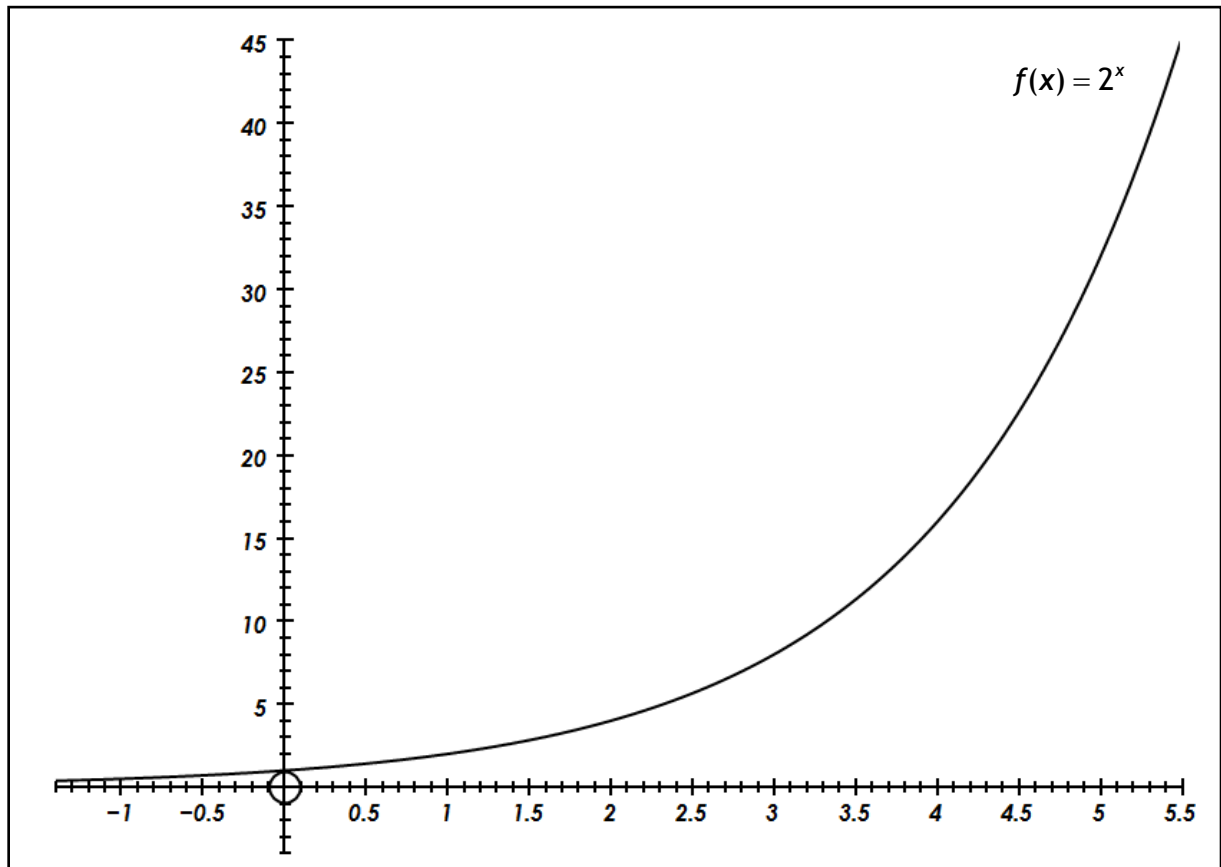
Use the graph to estimate:

(i) $\log_2(26)$

(ii) $\log_2(39.4)$

9. Drawing the graph of the inverse of $f(x) = 2^x$

(a) Fill in the table below and hence draw the graph of $g(x) = f^{-1}(x)$.

x	$f(x) = 2^x$	(x, y)
−2	$\dfrac{1}{4}$	$\left(-2, \dfrac{1}{4}\right)$
−1	$\dfrac{1}{2}$	$\left(-1, \dfrac{1}{2}\right)$
0	1	(0, 1)
1	2	(1, 2)
2	4	(2, 4)
3	8	(3, 8)

x	$g(x) = \log_2(x)$	(x, y)
$\dfrac{1}{4}$	−2	$\left(\dfrac{1}{4}, -2\right)$

(b) What is the relationship between $f(x) = 2^x$ and $g(x) = \log_2(x)$?

_____

(c) Explain why the relation $g(x) = \log_2(x), x \in \mathbb{R}^+$ is a function.

_____

_____

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(d) For $g(x) = \log_2(x)$:

    (i) Identify the base of $g(x) = \log_2(x)$.

    (ii) What is varying for the function $g(x) = \log_2(x)$.

    (iii) What is constant for the function $g(x) = \log_2(x)$.

(e) For $g(x) = \log_2(x)$:

    (i) What is the domain?

    (ii) What is the range?

(f) In relation to the graph of $g(x) = \log_2(x)$:

    (i) Is it a straight line?

    (ii) Is y increasing or decreasing as x increases?

    (iii) Describe how the rate of change varies as x increases.

(g) For $g(x) = \log_2(x)$:

    (i) Where does the graph cross the x-axis?

    (ii) What happens to the output as x decreases between 0 and 1?

    (iii) What is the y-intercept of the graph of $g(x) = \log_2(x)$.

    (iv) What is the relationship between the y-axis and the graph of $g(x) = \log_2(x)$.

10. Using the graph below, sketch and label the graphs of the following functions:
$h(x) = 10^x$, $k(x) = \log_{10}(x)$, $l(x) = e^x$ and $m(x) = \ln(x)$.

$f(x) = 2^x$

$g(x) = \log_2(x)$
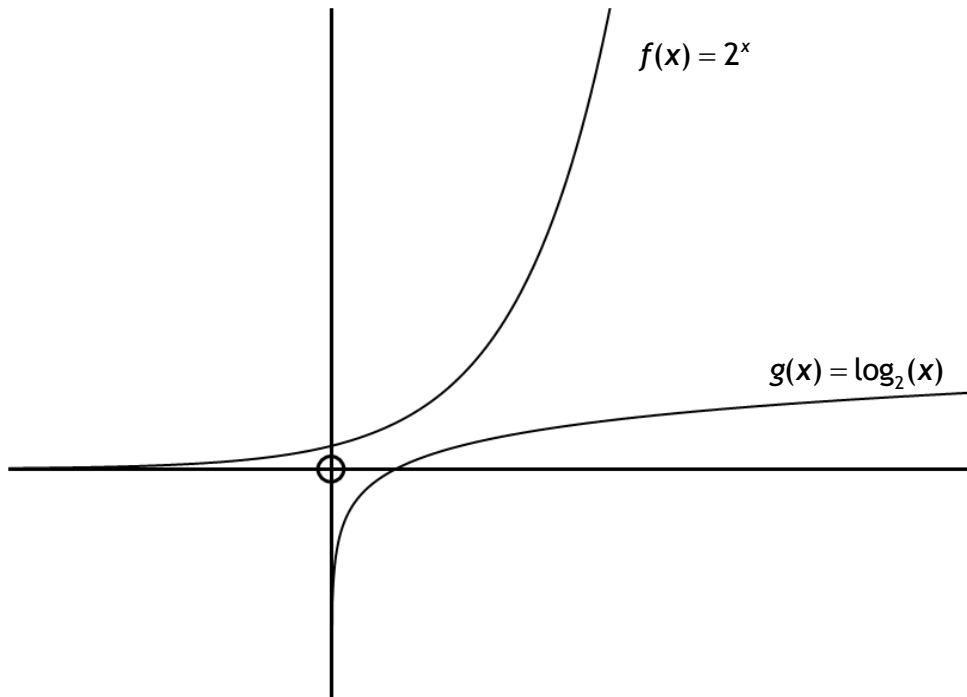
11. True or false discussion:

Equation	Equivalent exponential form	T/F	Correct equation (if false)
$\log_2 8 = 4$			
$\log_3 81 = 4$			
$\log_{10} 5 + \log_{10} 10 = \log_{10} 15$			
$\log_2 64 - \log_2 4 = \log_2 16$			
$\log_3\left(\dfrac{1}{81}\right) = -4$			
$2\log_2 8 = \log_2 16$			
$\log_2 4 + \log_2 128 = \log_2 512$			

12. Give possible numbers or variables for the blanks in the equations below.

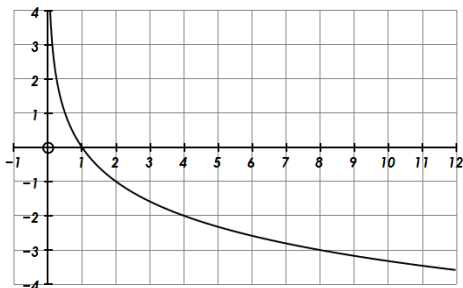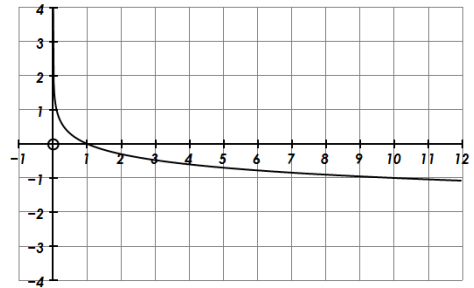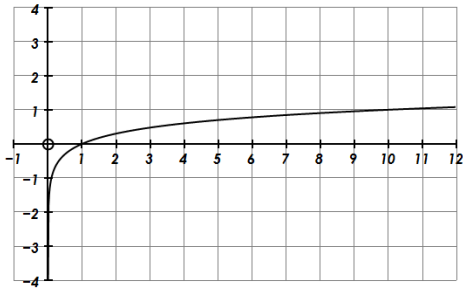|	
(i) $\log_\square \square = 3$	**(vi)** $\log_\square \square - \log_\square \square = 3$
(ii) $\log_\square \square = \square$	**(vii)** $\log_\square \square - \log_\square \square = \square$
	(viii) $\log_\square \square - \log_\square \square = \log_\square \square$
(iii) $\log_\square \square + \log_\square \square = 7$	**(ix)** $\square \log_\square \square = \log_\square \square$
(iv) $\log_\square \square + \log_\square \square = \square$	
(v) $\log_\square \square + \log_\square \square = \log_\square \square$	

13. **(i)** What is the relationship between $\log_b(x)$ and $\log_{\frac{1}{b}}(x)$?

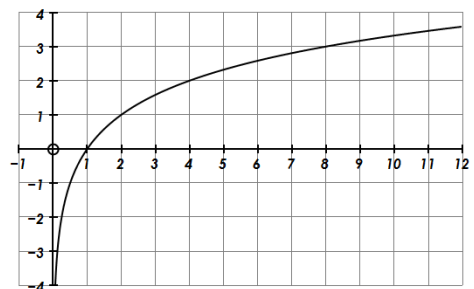**(ii)** Cut out the following and match each graph to its function.
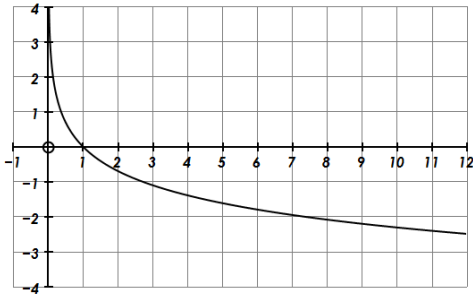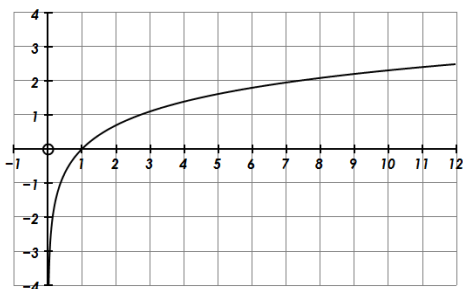
$j(x) = \log_{\frac{1}{10}}(x)$	$w(x) = \log_{10}(x)$
	$n(x) = \ln(x)$
	$p(x) = \log_2(x)$
$r(x) = \log_{\frac{1}{2}}(x)$	$u(x) = \log_{\frac{1}{e}}(x)$

14. **LCFL 2012 Paper 1 Q7 (b)**

A scientist is growing bacteria in a dish. The number of bacteria starts at 10 000 and doubles every hour.

(i) Complete the table below to show the number of bacteria over the next five hours

Time in hours	0	1	2	3	4	5
Number of bacteria (in thousands)	10					

(ii) Draw a graph to show the number of bacteria over the five hours.

(iii) Use your graph to estimate the number of bacteria in the dish after $2\frac{1}{2}$ hours.

(iv) The scientist is growing the bacteria in order to do an experiment. She needs at least 250 000 bacteria in the dish to do the experiment. She started growing the bacteria at 10:00 in the morning. At what time is the dish of bacteria ready for the experiment?

15. M_1 and M_2 are the magnitudes of two earthquakes on the Richter magnitude scale. If A_1 and A_2 are their corresponding amplitudes, measured at equal distances from the earthquakes, then $M_1 - M_2 = \log_{10}\left(\dfrac{A_1}{A_2}\right)$.

An earthquake rated 6.3 on the Richter magnitude scale in Iran on 26$^{\text{th}}$ Dec 2003 killed 40,000 people. The earthquake on Banda Aceh on 26$^{\text{th}}$ Dec 2004 was rated at 9.2 on the Richter magnitude scale.

How many times greater in amplitude of ground motion was the earthquake in Banda Aceh compared to the earthquake on Iran?

16. *pH* is a measure of the acidity. The *pH* is defined as follows:

$pH = -\log_{10}[H^+]$ where $[H^+]$ is the hydrogen ion concentration in an aqueous solution.

A *pH* of 7 is considered neutral. For bases: $pH > 7$. For acids: $pH < 7$.   [at $25\,^{\circ}\text{C}$]

A substance has $[H^+] = 2.7 \times 10^{-5}$ moles/litre.

Determine the pH and classify the substance as an acid or a base.

17. Sound levels are measured in decibels (dB).

If we are comparing two sound levels, B_1 and B_2 measured in dB,

$$B_2 - B_1 = 10\log_{10}\left(\dfrac{I_2}{I_1}\right)$$

The sound levels at The Who concert in 1976 at a distance of 46 m in front of the speakers was measured as B_2 = 120 db. What is the ratio of the intensity I_2 of the band sound at that spot compared to a jackhammer operating at a sound level of B_1 = 92 db at the same spot.

18. €140,000 was deposited at the beginning of January 2005 into an account earning 7% compound interest annually.
When will the investment be worth €200,000?

Verify and justify that the following two formulae give the same answer.
$$F = 140,000(1.07)^t$$
$$F = 140,000e^{0.0676586485t}$$
Comment on that answer in each case.