1.06 Circle Connections Plan

The first two pages of this document show a suggested sequence of teaching to emphasise the connections between synthetic geometry, co-ordinate geometry (which connects algebra and synthetic geometry) and trigonometry, using the circle as the unifying theme. Further connections can be made to complex numbers e.g. multiplication by i, modulus of a complex number and polar form of a complex number but these are not dealt with specifically in this document. The remainder of the document (pages 15 to 25) looks at each part of the sequence in more detail.

Part A

1. Terms and definitions associated with the circle

2. Transformations of the circle

3. Lengths and areas associated with the circle including such questions as:

    (i) What happens to the circumference if the radius length is doubled?

    (ii) What happens to the area if the radius length is doubled?

4. Angle measures in the circle: degrees and radians

Part B

This part connects synthetic and co-ordinate geometry. However, the students just go as far as finding the centre and radius length of the circle in Part B, and do not derive the equation of the circle until Part C. The equation of the circle, which may appear new to students at this stage, is left until later, in order to emphasise the connections to geometry first. It is a very small step to take, from knowing the centre and radius of a circle, to finding its equation. Finding the centre and radius, given various clues, is where most of the thinking occurs. Deriving the equation of the circle is yet another example of where students can themselves construct new knowledge from prior knowledge.

The theorems and constructions associated with the circle, are listed below and later on (pages 15 to 25) more detail is given on connecting them to the coordinated plane. The intention is that students will move seamlessly between the uncoordinated and the coordinated plane, verifying geometric results using algebraic methods. For example, students could construct a tangent to a circle at a point on an uncoordinated plane. This could be followed by constructing a tangent to a circle with a given centre, at a given point of contact on the coordinated plane. Using algebraic methods, the equation of the tangent is found and plotted, thus verifying the construction of the tangent.

Note: It is envisaged that students would have a list of the theorems and constructions available to them to facilitate making connections.

(1) Theorem 19, Corollary 5

Theorem 19: The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.

Corollary 5: If ABCD is a cyclic quadrilateral, then opposite angles sum to 180°, (and converse).

(2a) Theorem 20, Construction 19 and Corollary 6

Theorem 20: (i) Each tangent is perpendicular to the radius that goes to the point of contact.

(ii) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is a tangent to s.

Construction 19: Tangent to a given circle at a given point on the circle

Corollary 6: If two circles share a common tangent line at one point, then the two centres and that point are collinear.

(2b) Theorem 20, Construction 1 and Construction 17

Theorem 20: (i) Each tangent is perpendicular to the radius that goes to the point of contact .

(ii) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is a tangent to s.

Construction 1: Bisector of a given angle, using only compass and straight-edge.

Construction 17: Incentre and incircle of a given triangle, using only straight-edge and compass.

(3) Constructions 2 & 16, Corollaries 3 & 4 and Theorem 21

Construction 2: Perpendicular bisector of a segment, using only compass and straight-edge.

Construction 16: Circumcentre and circumcircle of a given triangle, using only straight edge and compass.

Corollary 3: Each angle in a semi-circle is a right angle.

Corollary 4: If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.

Theorem 21: (i) The perpendicular from the centre to a chord bisects the chord.

(ii) The perpendicular bisector of a chord passes through the centre.

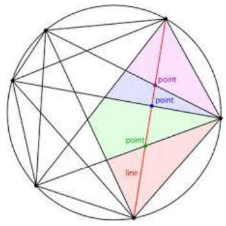
Part C

1. The equation of the circle derived by students using prior knowledge
2. Two equivalent forms of the circle equation

Part D

1. The unit circle and the trigonometric functions for all angles
2. The graphs of $f(x) = a + bsin(cx)$ and $g(x) = a + bcos(cx)$

Part A



Initial Vocabulary and definitions

Circle, radius, diameter, chord, tangent, point of contact, circumference, arc, semicircle, disc, sector, segment, angle at the centre of a circle standing on an arc or standing on a chord (central angle), quadrants of a circle etc. (Note: When using the word radius we are using it as a label but when we want to refer to the length of the line segment labelled radius, we use the phrase "length of the radius".)

New vocabulary, such as radian, circumcircle, incircle etc. will arise as new knowledge is acquired.

Activity: Tarsias on circle vocabulary (available on www.projectmaths.ie)
Apart from the usual way of using the Tarsias i.e. matching terms and definitions or terms and diagrams, students could be asked to explain terms to each other. Working in pairs, one student explains the term and the other student draws a sketch of it. Students swap roles alternately.

Construction of a Circle

Connection: *Always linked to the definition*

- Construct a circle on a blank page.
  Construct a circle on a coordinate plane.
  What do you need to know in order to construct a unique circle?
- What changes about a circle if it is translated, undergoes central or axial symmetry or rotation?
  What stays the same?

These tasks can be executed on the coordinate plane i.e. given a centre and radius of a circle, find the new centre and radius under the given transformation.

Lengths associated with the circle

Connection: *Proportional reasoning*

Lengths: radius length, diameter length, length of the circumference, length of a semicircle, arc length

Students might collect and graph data for the circumference length C and the diameter length d of a set of different-sized circles. They could extend their prior knowledge of algebra and data analysis by recognising that the points (C, d) nearly form a straight line, so C/d is between 3.1 and 3.2—a rough estimation of π.

Connection: *Measurement, data analysis, rate of change, linear functions*

Assessment & Problem Solving: 2013 LCHL Q9 Paper 2 (Arbelos)

Connection: *Perimeter, patterns & generalising*

Areas associated with the circle

Connection: *Proportional reasoning*

Area of a disc, area of a sector, area of a segment

Assessment & Problem Solving: 2014 LCOL Q5 Paper 2 (area of sectors, decomposing areas)

Connection: *Area of a triangle*

Assessment & Problem Solving: 2013 LCHL Q9 Paper 2 (Arbelos)

Connection: *Pythagoras' Theorem, area, growing patterns and generalising*

Assessment & Problem Solving: 2014 LCHL Q9 Paper 2 (Lunes)

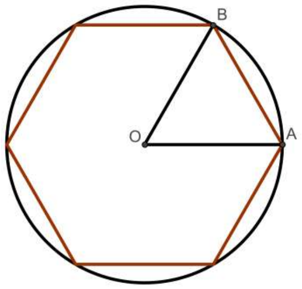
Connection: *Pythagoras' Theorem*

Angles in a circle (radian section is LCHL)

Connection: *Construction 18 (angle of 60° for LCOL and LCHL)*

Connection: *Construction of a hexagon in a circle as a possible extension*

Degree measure and radian measure



- Construct an angle of 60° in a circle.
  (Extension: Construct an angle of 30° to link to Construction 1: Bisector of a given angle)
- Construct a hexagon in a circle. (Start by drawing a line and mark a point on the line to be the centre of the circle.)
- How many 60 degree angles are there in a circle?
- If you were to measure a radius length *along the arc* AB, would the angle at the centre, subtended by the radius length, be less than or greater than 60°?
- This angle at the centre subtended by an arc equal in length to the radius is called a radian.
- How many radius lengths are in a circumference of a circle?

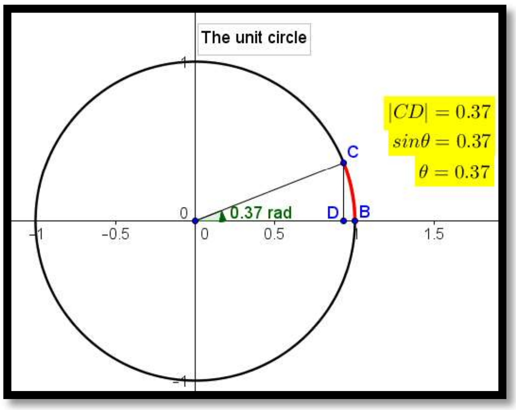
Activity for students supported by https://vine.co/v/MUqhTIOihrg (6 second videos)

Teaching and Learning Plan on radians: *www.projectmaths.ie*

Each radius length as an arc of a circle corresponds to a unique angle at the centre. We call this angle a **radian**. Note that we are now measuring angles in terms of lengths, which are real numbers.

(**The appeal of radian measure** lies in the way it enables use of linear measurement using real numbers for angles.)

Connection: *Limits: Students can intuitively see that* $\sin\theta$ *is approximately equal to* θ *in radians when theta is small. A GeoGebra file such as is shown below will help students to visualise this.*



- Convert degrees to radians and vice versa
- Find the length of an arc using radian measure. This follows from the definition of the radian.

Connection: *Connect to the method of finding arc length using degrees.*

- Derive the formula for the area of a sector using radian measure.

Connection: *Connect to the method for finding area of a sector using degrees.*

Part B

(1) Theorem 19 and corollaries 2 and 5 (LCHL and JCHL)

Theorem 19: The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.

Corollary 2: All angles at points of a circle, standing on the same arc, are equal.

Corollary 5: If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to 180°.

Activity for students: Constructing, investigating through measurement, use of dynamic geometry software, proof)

Assessment & Problem Solving: 2012 LCHL Q6 (b) Paper 2 (Cyclic quadrilateral)

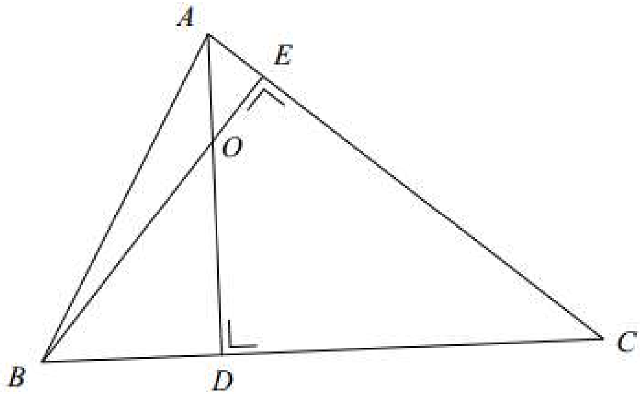
Question 6B

ABC is a triangle.

D is the point on BC such that $AD \perp BC$.
E is the point on AC such that $BE \perp AC$.

AD and BE intersect at O.

Prove that $|\angle DOC| = |\angle DEC|$.



Assessment & Problem Solving: 2008 LCHL Q1 (c) (ii) Paper 2 below shows an application of Theorem 19.

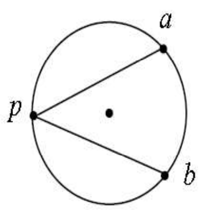
Given that at this point in the sequence, students have not yet dealt with the equation of a circle, this question can be modified to ask just for the centre and radius of the circle for part (i)

The connection to Theorem 19 is in part (ii).

(c) A circle passes through the points $a(8, 5)$ and $b(9, -2)$.
The centre of the circle lies on the line $2x - 3y - 7 = 0$.

(i) Find the equation of the circle.

(ii) p is a point on the major arc ab of the circle.
Show that $|\angle apb| = 45°$.



(2a) Theorem 20 & Construction 19 and Corollary 6

- A discussion of construction 19 for the tangent to a circle at a point, carried out (i) on an un-coordinated plane and (ii) on a co-ordinated plane, where the centre of the circle and the point of contact are given.
- The equation of the tangent can be found using algebraic methods. This tangent line can be plotted, thus verifying the construction.

Theorem 20: (i) Each tangent is perpendicular to the radius that goes to the point of contact.

(ii) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is a tangent to s.

Construction 19: Tangent to a given circle at a given point on it.

Extension: Link the construction of the tangent to the construction of the perpendicular bisector of a line segment.

- Construct the tangent to a circle centre (1,2) at a point (3,4) on the circle **on the coordinate plane** and find the equation of the tangent

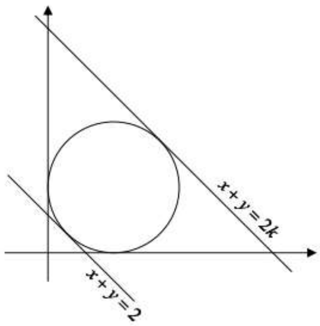
Assessment & Problem Solving: 2012 LCHL Q3 Paper 2 (Various solution methods are possible)

Connection: *Pythagoras' theorem, slopes of lines, tangents.*

Question 3 **(25 marks)**

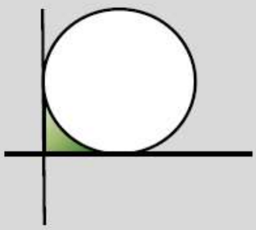
The circle shown in the diagram has, as tangents, the x-axis, the y-axis, the line $x + y = 2$ and the line $x + y = 2k$, where $k > 1$.

Find the value of k.

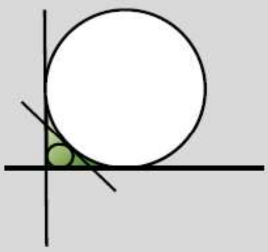


Assessment & Problem Solving: UL 2012 problem solving questions:

(a) The circle shown below has radius length r. What is the area of the shaded region in terms of r?

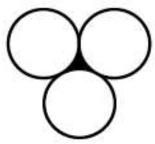


(b) Find the ratio of the areas of the two circles shown below

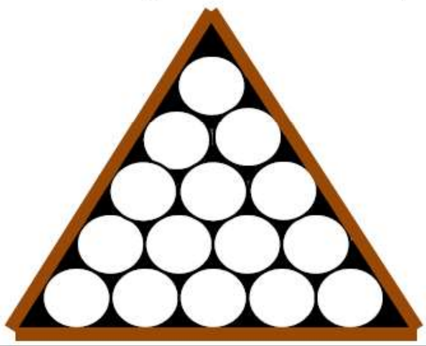


(c) Find the equation of the common tangent to both circles if the centre of the large circle is (4,4).

1 (a) In the diagram below the radius of each of the tree circles is r. Find in terms of r the radius of the shaded region between the circles.



(b) The diagram below represents 15 snooker balls each of radius r. Find in terms of r the area of the of the triangular frame which holds the balls in place.



The area referred to is the internal area of the triangular frame containing the snooker balls.

Corollary 6: If two circles share a common tangent line at one point, then the two centres and that point are collinear.

- What is the relationship between r_1 and r_2 when the circles touch externally/internally?

  <u>On the coordinate plane:</u>
  - Prove that two circles touch externally/internally given both centres and both radii.
  - Prove that two circles do not intersect given both centres and both radii.
  - If we know the point in common and both centres, find the equation of the common tangent.
  - If we know the centres, and the ratio of the lengths of the radii (if they touch externally), find the point of contact and the equation of the common tangent.
  - Given the centre and the equation of the tangent to a circle, find the radius of the circle. (Perp. Dis)

<u>Assessment & Problem Solving:</u> Given the centres and radii of two circles touching externally, find the point of contact of the two circles.

> ***Connection:*** *Proportional reasoning (e.g. 2014 Q4 LCHL Paper 2: Find the coordinates of the point of contact knowing the centres of the circles and lengths of the two radii without using a formula.)*

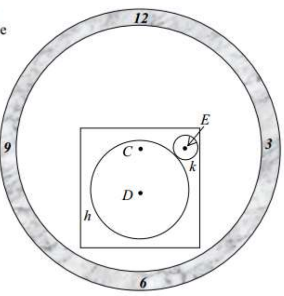
<u>Assessment & Problem Solving:</u> Questions showing the use of Corollary 6 on the coordinate plane.

2014 LCHL Q9 (a) (ii)

Question 9 **(60 marks)**

(a) The diagram shows a circular clock face, with the hands not shown. The square part of the clock face is glass so that the mechanism is visible. Two circular cogs, h and k, which touch externally are shown.

The point C is the centre of the clock face. The point D is the centre of the larger cog, h, and the point E is the centre of the smaller cog, k.



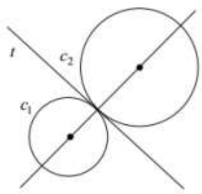
(i) In suitable co-ordinates, the equation of the circle h is
$$x^2 + y^2 + 4x + 6y - 19 = 0.$$
Find the radius of h, and the co-ordinates of its centre, D.

(ii) The point E has co-ordinates $(3, 2)$. Find the radius of the circle k.

2013 LCHL Q4: Find the point of contact and the equation of the tangent common to both circles:

Question 4 **(25 marks)**

The circles c_1 and c_2 touch externally as shown.



(a) Complete the following table:

Circle	Centre	Radius	Equation
c_1	$(-3, -2)$	2	
c_2			$x^2 + y^2 - 2x - 2y - 7 = 0$

(b) **(i)** Find the co-ordinates of the point of contact of c_1 and c_2.

(ii) Hence, or otherwise, find the equation of the tangent, t, common to c_1 and c_2.

2012 LCHL Q2: Prove that circles are touching given centres and radii, verify the point of contact, find the equation of the common tangent

2012 LCHL Sample Paper Q4: Using the perpendicular distance from the centre of the circle to the tangent being equal to the radius

2010 LCHL Q4: Find centre and radius given x-axis, $y = 6$ as tangents and the line $x - 2y - 1 = 0$ through the centre of the circle.

(2b) Construction 17 (incircle) using Theorem 20 and Construction 1

Theorem 20: (i) Each tangent is perpendicular to the radius that goes to the point of contact.

(ii) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is a tangent to s.

Construction 1: Bisector of a given angle, using only compass and a straight-edge

Construction 17: Incentre and incircle of a given triangle using only straight-edge and compass

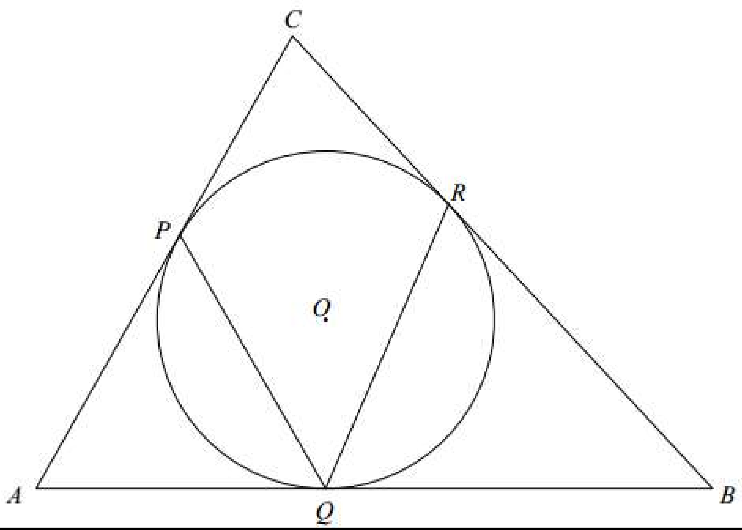
- Draw a sketch of the incircle of a triangle; the circle must touch the three sides of the triangle.
- What does the word "touch" indicate about the relationship between the sides of the triangle and the circle?
- What is the relationship between the sides of the triangle and the three radii of the incircle at the points of contact?
- Describe the distances between the centre of the incircle and the sides of the triangle at the point of contact given the definition of a circle.
- What construction will allow you to find the centre of the incircle?
- What information does an angle bisector give about the points on the angle bisector?
- Construct the bisectors of all the angles in a triangle.
- What do you notice about all the angle bisectors?
- Construct the incircle of the triangle.
- Construct the incircle of various triangles.

<u>Assessment & Problem Solving:</u> LCHL 2012 Q6B Paper 2 (incorporates Theorem 20, corollary 5, Theorem 19, Properties of an angle bisector)

Question 6B

The incircle of the triangle ABC has centre O and touches the sides at P, Q and R, as shown.

Prove that $|\angle PQR| = \frac{1}{2}\left(|\angle CAB| + |\angle CBA|\right)$.



This question (or the construction of an incircle) could be transferred to a coordinate plane.

(3) Construction 2, Construction 16, corollaries 3 and 4 and Theorem 21

- How many circles can be drawn through one point?
  How many circles can be drawn through two points? How are you constrained when given two points?

Connection: *Students should see that since the two points given have to be equidistant from one point i.e. the centre of the circle, that the construction of the perpendicular bisector of a line segment will give them the locus of all points equidistant from two points.*

- How many circles can be drawn through three non-collinear points? Can you build on the previous outcome? Carry out the construction.
- Could you have drawn more than one circle through those three points?
- How many points define a unique circle?
- Join up all three points to form a triangle.
- Draw the perpendicular bisector of the third side of the triangle constructed. What do you notice?
  (The 3 perpendicular bisectors pass through one point i.e. they are concurrent.
  New vocabulary: *Concurrent*)
- Using the list of constructions, which construction have you carried out?

(From the above activity the students have carried out **Construction 16**: Circumcentre and circumcircle of a given triangle using only compass and straight-edge. Ask students to label the circumcircle, circumcentre and circumradius on their constructions.)

- The sides of the triangle are chords of the circumcircle. What do you notice about the relationship between the chords of this circle, the perpendicular bisectors of the chords and the centre of the circle?
- Using the list of theorems, can you identify a theorem associated with your construction?

(Students now see an instance of **Theorem 21**: (1) The perpendicular from the centre of the circle to a chord bisects the chord. (2) The perpendicular bisector of a chord passes through the centre.)

- Construct the circumcircle for different types of triangles. Carry out the constructions with given points **on the coordinate plane** and verify the coordinates of the circumcentre and the length of the circumradius using algebra.
  Examples of triangles with integer coordinates:
  (1) Triangle with vertices H(-1,-3), I(5,9) and J(11,1) Acute
  (2) Triangle with vertices D(1,6), E(10,3) and F(2,3) Obtuse
  (3) Triangle with vertices A(1,3), B(7,5) and C(9,-1) Right angled
  (Different groups of students present their work (possibly using a visualizer) and patterns are identified.

- Verify Corollaries 3 & 4 **on the coordinate plane** using slopes and or/distances with triangle ABC
  Corollary 3: Each angle in a semicircle is a right angle. **Corollary 4**: If the angle standing on a chord [BC] at some point of the circle is a right angle, then [BC] is a diameter of the circle.

- Given 2 points $(8, 5)$ and $(9, -2)$ on a circle and the equation of a line $2x - 3y - 7 = 0$ through the centre of the circle, find the centre and radius of the circle.

Assessment & Problem Solving: LCHL 2013 Q9 (d) (Arbelos and cyclic quadrilaterals)

Assessment & Problem Solving: Workshop 6 questions

Part C

Equation of a circle

Students should understand that the derivation of the formula for the equation of a circle is based on Pythagoras' theorem.

If students have been exposed to deriving the **formula for the distance between two points**, they will see that the equation of the circle is just a statement of Pythagoras' theorem.

We could start with a circle with centre (0,0) as shown first below, or we could start with a circle with centre not $(0,0)$ and show that the equation of a circle with centre $(0,0)$ is a special case of the equation of a circle with centre (h, k).

Note: To derive the equation of the circle, one could begin with a circle with centre $(0,0)$ passing through a specific point and then generalise to find the equation of a circle with centre (h, k) and passing through the point (x, y). The following sequence (i) to (iv) however, begins with a circle passing through a centre which is not (0,0) and in part (iii) we derive the equation of a circle with centre (h, k) and passing through a point (x, y). We then show that the equation of the circle with centre (0,0) passing through a point (x, y) is a special case where (h, k) = $(0,0)$. Both approaches help students make sense of the equation of the circle formula.

Part (i): Circle with centre $O(1, 2)$ and containing the point $A(5, 7)$ *(starting with a specific circle)*

- Construct a line segment from $A(5,7)$ to $B(5,2)$ and a line segment from $O(1,2)$ to $B(5,2)$.
- Complete the right triangle using a line segment from $(1,2)$ to $(5,7)$ as the hypotenuse, of length r, of the triangle
- Mark in the lengths of the other two sides of the right triangle.
- Write down Pythagoras' Theorem for this triangle and use it to find the radius length r.
- Find the distance r from (1,2) to $(5,7)$ using the distance formula and compare with the result using Pythagoras' Theorem.

Part (ii): Using a general point (x, y) on any circle with centre (1, 2)

- Construct a circle, centre $O(1,2)$ passing through a point A(x, y).
- Construct line segments from $A(x, y)$ to $B(x, 2)$ and from $O(1,2)$ to $B(x, 2)$.
- Construct a right triangle using a line segment from $(1, 2)$ to (x, y) as the hypotenuse of the triangle of length r.
- Mark in the lengths of the other two sides of the right triangle in terms of x and y.
- Write down Pythagoras' Theorem for this triangle and use it to find the radius length r.
- Find the distance r from (1,2) to (x, y) using the distance formula and compare with the equation for r using Pythagoras' Theorem.

Part (iii): A circle with any centre $O(h,k)$ and any point $A(x,y)$ on its circumference

- Construct a circle, centre $O(h,k)$ passing through a point $A(x,y)$.
- Construct line segments from $A(x,y)$ to $B(x,k)$ and from $O(h,k)$ to $B(x,k)$.
- Construct a right triangle using a line segment from $(h,\ k)$ to (x,y) as the hypotenuse of length r, of the triangle
- Mark in the lengths of the other two sides of the right triangle in terms of $x,\ y,\ h$ and k.
- Write down Pythagoras' Theorem for this triangle and use it to find the radius length r.
- Find the distance r from (x,y) to (h,k) using the distance formula and compare with the result using Pythagoras' Theorem.
- Is this equation true for all circles? (Go back to the definition of the circle.)

Part (iv): Showing that the equation of circle with centre $(0,0)$ and radius length r is a special case of the equation of a circle with centre (h,k) and radius length r.

- Given the equation of the circle $(x-h)^2+(y-k)^2=r^2$, what would be the equation of a circle with centre $(0,0)$ and radius length r?

Follow up:

- Use the equation of the circle formula to get the equations of circles with different centres and different radii
- Write down circle equations and then find their centres and radii
- Use two ways of checking if a point is on, inside, or outside the circle. Compare both ways.
- Given a circle with a radius which is the largest number of a Pythagorean triple, use the particular Pythagorean triple to find 12 points on the circle with integer values.

Equivalent forms of the equation of the circle

- If $x^2 + y^2 = r^2$ is the equation of a circle, centre $(0,0)$ then $mx^2 + my^2 = mr^2$, $m\epsilon R$, is an equivalent form of the equation of that circle. We will look at other equivalent forms of the equation of a circle.
- Expand the equation of the circle, given in the form $(x - h)^2 + (y - k)^2 = r^2$ and show that the expanded form is equivalent to $x^2 + y^2 + 2gx + 2fy + c = 0$
- Explain where c comes from and how it can be used to find r . Explain the relationship between h, k, g and f by comparing coefficients. Given the circle of the equation in the form $x^2 + y^2 + 2gx + 2fy + c = 0$, how can you find the centre and radius of the circle?
- Starting with the form $x^2 + y^2 + 2gx + 2fy + c = 0$ of the equation of the circle, practice the procedure of completing the square to write the equation of the circle in the form $(x - h)^2 + (y - k)^2 = r^2$.
- $mx^2 + my^2 + 2gmx + 2fmy + mc = 0$, $m\epsilon R$, is an equivalent form of $x^2 + y^2 + 2gx + 2fy + c = 0$. Given $mx^2 + my^2 + 2gmx + 2fmy + mc = 0$, how can you find the centre and radius of the circle?
- Given either form of the equation of the circle, $(x - h)^2 + (y - k)^2 = r^2$ or $x^2 + y^2 + 2gx + 2fy + c = 0$, derive the other equivalent form.

Further work with both forms of the equation of a circle

- How do you recognise the equation of a circle? What order is the equation of a circle? Does it have an xy term? Compare the coefficients of the x^2 and y^2 terms in the circle equation.
- Do the following equations represent the equations of circles:
  $3x^2 + y^2 = 9$? $x^2 + (y + 5)^2 = 1$? Explain your answers.
- What type of circle cannot pass through the origin? (Circle with centre $(0,0)$.)
- Given the equation of a circle as $x^2 + y^2 + 2gx + 2fy + c = 0$, how would you know if the circle passed through the origin?
- Write down your own circle equations (in the two equivalent forms) and then find the centres and radii of these circles. Work backwards to verify your answers.
  Verify using GeoGebra also.
- How could we find the equation of a circle in two different ways if given 3 points on the circle?
- What is the equation of a circle with centre (h, k) and radius r, if its centre is on the x–axis?
- What is the equation of a circle with centre (h, k) and radius r, if its centre is on the y–axis?
- What is the equation of a circle which touches the x-axis?
- What is the equation of a circle which touches the y-axis?
- What is the equation of a circle which touches both the x- axis and the y-axis?

Assessment & Problem Solving: 2014, Q4, Paper 2 Sample Paper.

Assessment & Problem Solving: 2011 Q5, Paper 2 (intersection of lines and circles).
(Intersection of lines and circles: Students can learn for the first time about solving between a quadratic equation and a linear equation in the context of coordinate geometry of the circle. This is an opportunity to review key skills in algebra. Once the procedure has been learned for finding the point of intersection between a line and a circle, the students can easily progress to finding the point of intersection between a line and other shapes described by equations of degree 2.)

Assessment & Problem Solving: Find the length of the chord cut off the line $x + y - 5 = 0$ by the circle $x^2 + y^2 = 13$ in more than one way.

Part D

Unit circle and the trigonometric functions

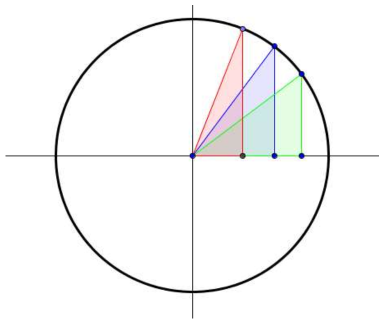
- Construct a circle with centre $(0,0)$ and radius 1 on the coordinate plane.
  This is called the ***unit circle*** *as it has centre* $(0,0)$ *and radius equal to one unit.*
- Name 4 points with integer coordinates on the unit circle.
- Choose any point with coordinates (x, y) on the unit circle, in the first quadrant.
  Draw in the radius from $(0,0)$ to (x, y).
- Draw a line segment from (x, y) to $(x, 0)$. Draw a line segment from $(0,0)$ to $(x, 0)$.
  This completes the right angled triangle with vertices $(0,0)$, $(x, 0)$ and (x, y).
- Write down Pythagoras' Theorem for this right angled triangle.
  Alternatively use the distance formula to find the distance from $(0,0)$ to (x, y). Rewrite the equation without square roots.

$$x^2 + y^2 = 1$$

- Mark in the angle θ in standard position in the triangle drawn in the first quadrant.
- Write down the relationship between $\sin\theta$ and the y-coordinate of where the terminal ray of the angle θ intersects the unit circle. ($y = sin\theta$)
- Write down the relationship between $\cos\theta$ and the x-coordinate of where the terminal ray of the angle intersects the unit circle. ($x = cos\theta$)

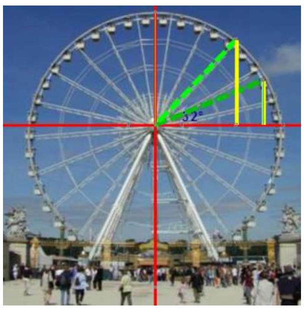
The fact that $x = cos\theta, y = sin\theta$ is true for all values of θ in the unit circle enables us to extend our use of the trigonometric ratios (learned in Junior Certificate and restricted to angles between $0°$ and $90°$) to trigonometric functions which model real life phenomena that occur in cycles.

Triangles occur in circles. Circles link to cycles. Trigonometric functions can be used to model real life phenomena that occur in cycles.



- Prove that $sin^2\theta + cos^2\theta = 1$.

Using the real-life analogy with movement of a point on a Ferris wheel, describe how $sin\theta$ and $cos\theta$ vary as a point moves from $(1,0)$ around the wheel back to $(1,0)$. We are looking at co-variation here. What happens to the sine (or cosine) of the angle as we rotate around the coordinate grid? The sine (or cosine) value of the angle is a function of the angle of rotation. Sometimes we refer to these trigonometric functions as circular functions because they can be defined using the unit circle.



Plot graphs of $f(x) = sin(x)$ and $g(x) = cos(x)$ and compare. Plot graphs of $f(x) = a + b\ sin\ c(x)$. Relate a and b back to the circle.

Notes on Circle Connections:

1.07 Theorems and Corollaries

1. Vertically opposite angles are equal in measure.
2. In an isosceles triangle, the angles opposite the equal sides are equal, (and converse).
3. If a transversal makes equal alternate angles on two lines, then the lines are parallel, (and converse).
4. The angles in any triangle add to $180°$.
5. Two lines are parallel if and only if, for any transversal, corresponding angles are equal.
6. Each exterior angle of a triangle is equal to the sum of the interior opposite angles.
7. The angle opposite the greater of two sides is greater than the angle opposite the lesser side, (and converse).
8. Two sides of a triangle are together greater than the third.
9. In a parallelogram, opposite sides are equal, and opposite angles are equal, (and converses).

   Corollary 1: A diagonal divides a parallelogram into two congruent triangles.
10. The diagonals of a parallelogram bisect one another.
11. If three parallel lines cut off equal segments on some transversal line, then they will cut off equal segments on any other transversal.
12. Let ΔABC be a triangle. If a line l is parallel to BC and cuts $[AB]$ in the ratio $s:t$, then it also cuts $[AC]$ in the same ratio, (and converse).
13. If two triangles ΔABC and $\Delta A'B'C'$ are similar, then their sides are proportional, in order, (and converse).
14. In a right-angled triangle the square of the hypotenuse is the sum of the squares of the other two sides.
15. If the square of one side of a triangle is the sum of the squares of the other two, then the angle opposite the first side is a right angle.
16. For a triangle, base times height does not depend on choice of base.
17. A diagonal of a parallelogram bisects the area.
18. The area of a parallelogram is the base by the height.
19. The angle at the centre of a circle standing on a given arc is twice the angle at any point of the circle standing on the same arc.

   Corollary 2: All angles at points of a circle, standing on the same arc, are equal (and converse).

   Corollary 3: Each angle in a semi-circle is a right angle.

   Corollary 4: If the angle standing on a chord $[BC]$ at some point of the circle is a right angle, then $[BC]$ is a diameter.

   Corollary 5: If $ABCD$ is a cyclic quadrilateral, then opposite angles sum to $180°$,(and converse).
20. (i) Each tangent is perpendicular to the radius that goes to the point of contact.

   (ii) If P lies on the circle s, and a line l through P is perpendicular to the radius to P, then l is a tangent to s.

   Corollary 6: If two circles share a common tangent line at one point, then the two centres and that point are collinear.
21. (i) The perpendicular from the centre to a chord bisects the chord.

   (ii) The perpendicular bisector of a chord passes through the centre.

1.08 Constructions

1. Bisector of a given angle, using only compass and straight edge.
2. Perpendicular bisector of a segment, using only compass and straight edge.
3. Line perpendicular to a given line l, passing through a given point not on l.
4. Line perpendicular to a given line l, passing through a given point on l.
5. Line parallel to given line, through given point.
6. Division of a segment into $2, 3$ equal segments, without measuring it.
7. Division of a segment into any number of equal segments, without measuring it.
8. Line segment of given length on a given ray.
9. Angle of given number of degrees with a given ray as one arm.
10. Triangle, given lengths of three sides.
11. Triangle, given SAS data.
12. Triangle, given ASA data.
13. Right-angled triangle, given the length of the hypotenuse and one other side.
14. Right-angled triangle, given one side and one of the acute angles (several cases).
15. Rectangle, given side lengths.
16. Circumcentre and circumcircle of a given triangle, using only straight edge and compass.
17. Incentre and incircle of a given triangle, using only straight-edge and compass.
18. Angle of $60°$, without using a protractor or set square.
19. Tangent to a given circle at a given point on the circle.
20. Parallelogram, given the length of the sides and the measure of the angles.
21. Centroid of a triangle.
22. Orthocentre of a triangle.