Project Maths
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Development Team

Module 4

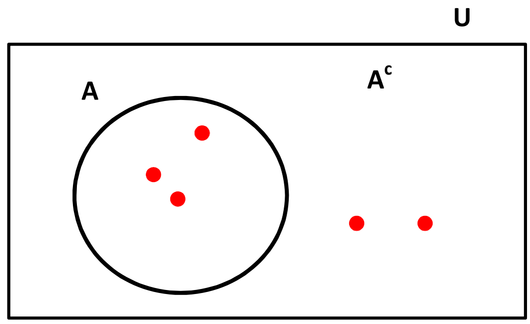
Set Theory and Probability

It is often said that the three basic rules of probability are:

1. Draw a picture
2. Draw a picture
3. Draw a picture

Venn diagrams are particularly useful when answering questions on conditional probability.

1. Complement Rule

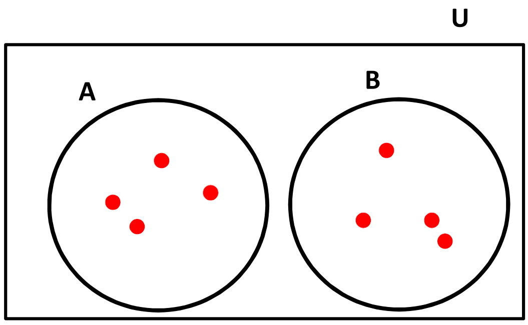


The probability of an event occurring is 1 − the probability that it does not occur.

$$P(A) = 1 - P(A^c)$$

2. Disjoint Sets (Mutually Exclusive Events)

Two events are said to be Mutually Exclusive if they have no outcomes in common.



For Mutually Exclusive events A and B, the probability that one or other occurs is the sum of the probabilities of the two events.

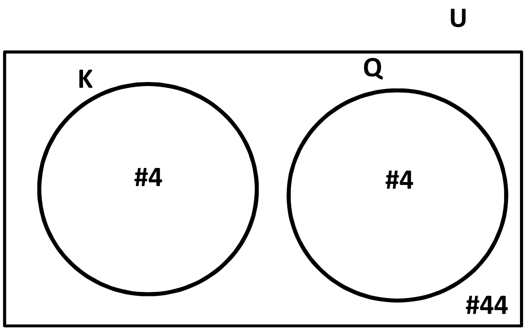
$$P(A \text{ or } B) = P(A) + P(B)$$

$$P(A \cup B) = P(A) + P(B)$$

provided that A and B are **mutually exclusive** events.

Mutually exclusive events are disjoint sets.

Example
A card is drawn from a pack of 52 cards. What is the probability that the card is either a Queen or a King?

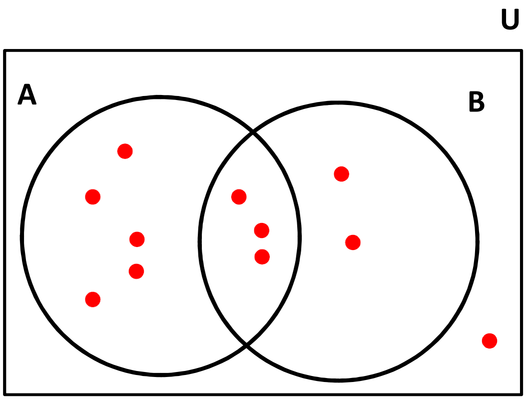


These events are Mutually exclusive "they have no outcome in common"

$$\text{Probability } P(Q \text{ or } K) = P(Q) + P(K) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

3. Overlapping Sets (Non Mutually Exclusive Events)

Two events are not Mutually Exclusive if they have outcomes in common.

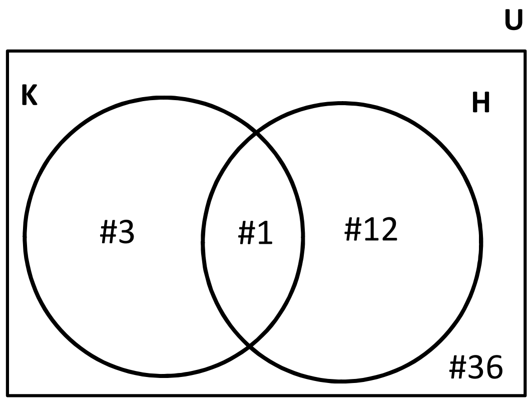


$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

A and B are **not mutually exclusive** events.

Example
A card is drawn from a pack of 52 cards. What is the probability that the card is either a King or a Heart?



These two events have one outcome in common i.e. "the king of hearts".

The two events are not "Mutually Exclusive" i.e. they have an outcome in common

$$\text{P(King or Heart)} = \text{P(King)} + \text{P(Heart)} - \text{P(King which is a Heart)}$$
$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52}$$
$$= \frac{16}{52}$$
$$= \frac{4}{13}$$

4. The Multiplication Law for Independent Events

Two events are said to be independent if one event does not affect the outcome of the other.

$$\left.\begin{aligned} P(A \text{ and } B) &= P(A)\times P(B) \\ P(A\cap B) &= P(A)\times P(B) \end{aligned}\right\} \text{ if A and B are } \textbf{independent events}$$

$$\left.\begin{aligned} P(A \text{ and } B) &= P(A)\times P(B\,|\,A) \\ P(A\cap B) &= P(A)\times P(B\,|\,A) \end{aligned}\right\} \text{ for all events}$$

NOTE: $P(B\,|\,A)$ is the probability of B given A has already occurred.

Example
A card is drawn from a pack of 52 cards and then replaced. A second card is then drawn from the pack. What is the probability that the two cards drawn are clubs?

Solution
These two events are independent as the outcome of drawing the second card is not affected by the outcome of drawing the first card because the first card was replaced.

$$\text{P(Club and Club)} = \text{P(Club)} \times \text{P(Club)}$$
$$= \frac{13}{52} \times \frac{13}{52}$$
$$= \frac{1}{16}$$

5. The Multiplication Law for Non Independent Events

Two events are not Independent if one event affects the outcome of the other.

$$P(A \text{ and } B) = P(A) \times P(B \mid A) \text{ for all events}$$

NOTE : $P(B \mid A)$ is the probability of B given A has already occured.

Example

A card is drawn from a pack of 52 cards. A second card is then drawn from the pack. What is the probability that the two cards drawn are clubs? (no replacement)

Solution

These two events are not independent as the outcome of drawing the second card is affected by the outcome of drawing the first card because the first card was not replaced.

$$P(\text{Club and Club}) = P(\text{Club}) \times P(\text{Second card is a club given that the first card was a club})$$

$$P(\text{Club and Club}) = P(\text{Club}) \times P(\text{Club} \mid \text{Club})$$

$$= \frac{13}{52} \times \frac{12}{51}$$

$$= \frac{1}{4} \times \frac{12}{51} = \frac{1}{17}$$

This is Conditional Probability: The probability of the second event is conditional on the first

Conditional Probability

Conditional Probability is the probability of an event which is affected by another event.

$P(A \text{ and } B) = P(A) \times P(B \mid A)$ if A and B are not independent events

This is often written as: $\boxed{P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}}$

This reads as:

"the probability of B given A equals the probability of A and B over the probability of A".

Example 1

A bag contains 9 identical discs, numbered from 1 to 9.

One disc is drawn from the bag.

Let A = the event that 'an odd number is drawn.'

Let B = the event 'a number less than 5 is drawn'

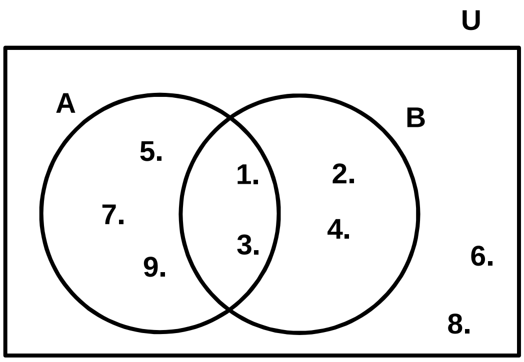
(i) What is the probability that the number drawn is less than 5 given that it is odd i.e. P(B|A)?

(ii) What is P(A|B)? Explain in words first of all what this questions means and then evaluate it.

(iii) Are the events A and B independent, Justify your answer?

Solution

(i) Students should work with a Venn diagram and then explain what they have done by formula.

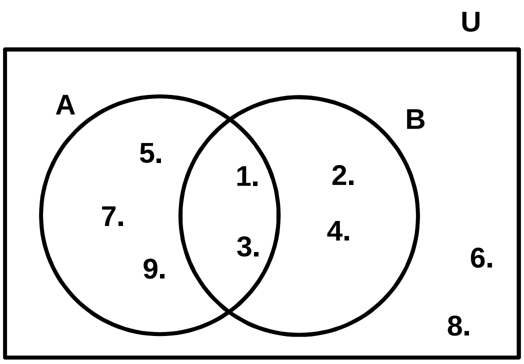


$$P(B|A) = \frac{\#(A\cap B)}{\#A} = \frac{2}{5}$$

More generally we can say

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{2/9}{5/9} = \frac{2}{5}$$

(ii) P(A|B) = probability that the number drawn is odd given that it is less than 5



$$P(A|B) = \frac{\#(A\cap B)}{\#B} = \frac{2}{4} = \frac{1}{2}$$

More generally we can say

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{2/9}{4/9} = \frac{1}{2}$$

(iii) No the events A and B are not independent.

Is $P(B) = P(B|A)$? or Is $P(A)\times P(B) = P(A\cap B)$?

$$P(B) = \frac{\#B}{\#U} = \frac{4}{9} \qquad P(A)\times P(B) = \frac{5}{9}\times\frac{4}{9} = \frac{20}{81}$$

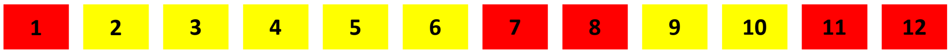
$$P(B|A) = \frac{2}{5} \qquad P(A\cap B) = \frac{2}{9}$$

Since either of these $\left(\text{i.e. } \frac{4}{9} \neq \frac{2}{5} \text{ or } \frac{20}{81} \neq \frac{2}{9}\right)$ are not equal the two events are not independent.

i.e. the fact that the event A happened changed the probability of the event B happening.

Example 2

A game is played with 12 cards, 5 of the cards are red {1, 7, 8, 11, 12} and 7 are yellow. The cards are numbered from 1 to 12.



(i) Given that a card is red what is the probability that the number on it is even?

(ii) Given that a card is red what is the probability that the number on it is odd? Work out this answer in 2 ways.

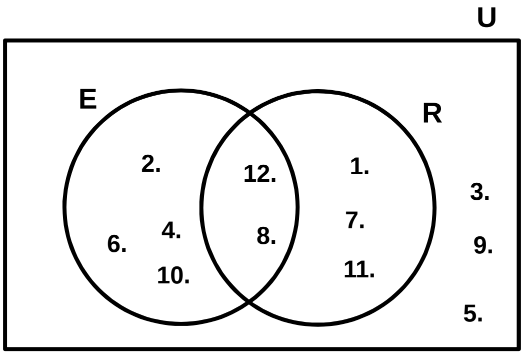
(iii) Given that the number on the card is even what is the probability that it is red?

(iv) Is P(E|R) = P(R|E) where E means card has an even number on it and R means that the card is red?

(v) Is P(E|R) = P(E)? What does your answer tell you about the events of being red and being even?

Solution

(i) P(E|R) = probability that the number drawn is even given that it is red.

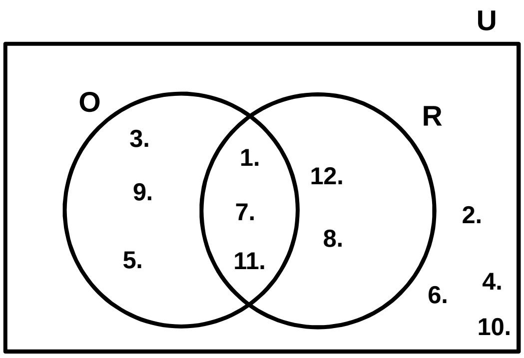


$$P(E|R)=\frac{2}{5}$$

More generally we can say

$$P(E|R)=\frac{P(E\cap R)}{P(R)}=\frac{2/12}{5/12}=\frac{2}{5}$$

(ii) P(O|R) = probability that the number drawn is odd given that it is red.



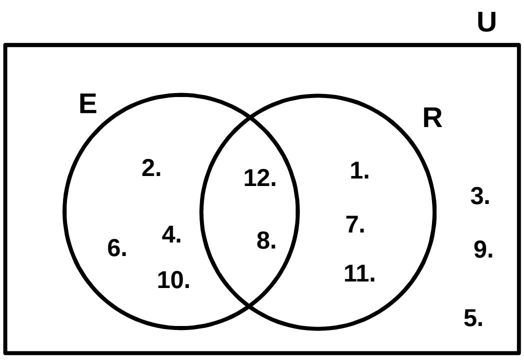
$$P(O|R)=\frac{3}{5}$$

More generally we can say

$$P(O|R)=\frac{P(O\cap R)}{P(R)}=\frac{3/12}{5/12}=\frac{3}{5}$$

We could also say that it is $1-\ P(E|R)=1-\frac{2}{5}\ =\frac{3}{5}$

(iii) P(R|E) = probability that the number drawn is red given that it is even.



$$P(R|E)=\frac{2}{6}$$

More generally we can say

$$P(R|E)=\frac{P(R\cap E)}{P(E)}=\frac{2/12}{6/12}=\frac{1}{3}$$

(iv) Is $P(E|R) = P(R|E)$?

$P(E|R) = \frac{2}{5}$ from part (i)

$P(R|E) = \frac{1}{3}$ from part (iii)

$\therefore$ they are not equal

(v) Is $P(E|R) = P(E)$?

What does your answer tell you about the events of being red and being even?

$P(E|R) = \frac{2}{5}$ from part (i)

$P(E) = \frac{1}{2}$

$\therefore$ they are not equal

Hence the events E and R are not independent.

Landlines versus Mobiles

According to estimates from the federal government's 2003 National Health Interview Survey, based on face-to-face interviews in 16,677 households, approximately 58.2% of U.S. adults have both a land line and a mobile phone, 2.8% have only mobile phone service, but no landline, and 1.6% have no telephone service at all.

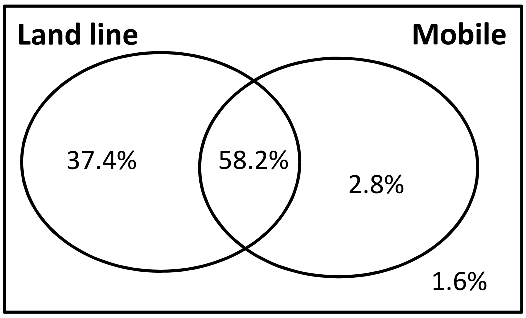
(a) What proportion of U.S. households can be reached by a landline call?

(b) Are having a mobile phone and a having a landline independent events? Explain.

Solution:

(a) Since 2.8% of U.S. adults have only a mobile phone, and 1.6% have no phone at all, polling organisations can reach 100 – 2.8 – 1.6 = 95.6% of U.S. adults.

(b) Using the Venn diagram, about 95.6% of U.S. adults have a landline. The probability of a U.S. adult having a land line given that they have a mobile phone is 58.2/(58.2+2.8) or about 95.4%. **It appears** that having a mobile phone and having a land line are **independent**, since the probabilities are roughly the same.



$P(L) = 95.6\%$

$$P(L|M) = \frac{58.2}{58.2 + 2.8} \approx 95.4\%$$

Taken from : Stats, Data and Models (2nd Ed.), pg. 387

Expected Value

Random Variable

The outcome of an experiment need not be a number, for example, the outcome when a coin is tossed can be 'heads' or 'tails'. However, we often want to represent outcomes as numbers. A random variable is a function that associates a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial to trial as the experiment is repeated.

There are two types of random variable:

1. Discrete **2.** Continuous

Examples

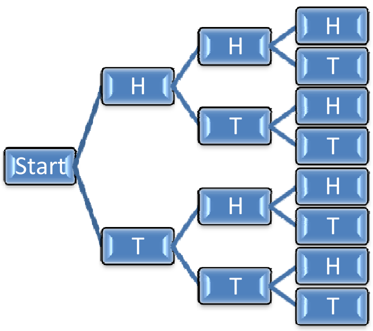
1. *A coin is tossed ten times. The random variable X is the number of tails that are noted. X can only take the values 0, 1, ..., 10, so X is a discrete random variable.*

2. *A light bulb is burned until it burns out. The random variable Y is its lifetime . Y can take any positive real value, so Y is a continuous random variable.*

A random variable has a probability distribution i.e. an assignment of probabilities to the specific values of the random variable or to a range of its values. A discrete random variable gives rise to a discrete probability distribution and a continuous random variable gives rise to a continuous probability distribution.

Example 1

If I toss 3 coins. x = the number of heads I get.



H	H	H	3 heads
H	T	H	2 heads
H	T	H	2 heads
H	T	T	1 head
T	H	H	2 heads
T	H	T	1 head
T	T	H	1 head
T	T	T	0 heads

x	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

This is a probability distribution. It is similar to a frequency distribution.

We calculate mean and standard deviation for a probability distribution in the same way we calculated the mean and standard deviation of a frequency distribution.

The mean of the probability distribution $\mu = \dfrac{\sum xP(x)}{\sum P(x)}$

As $\sum P(x) = 1 \quad \mu = \sum xP(x)$.

This is called the **EXPECTED VALUE**.

$$\mu = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right) = \frac{12}{8} = 1.5$$

Total of 8 outcomes

Example 2

If the above example represented a game at a carnival where you tossed 3 coins together and you win 2 euro for each head which shows you could get 0, 1, 2 or 3 heads and hence win 0,2, 4, or 6 euro.

We consider 3 headings: **(i)** the outcome

(ii) the probability associated with each outcome

(iii) the value associated with each outcome (the random variable)

Outcome (no. of heads)	Probability of each outcome P(x)	Value associated with each outcome (in euro) (x)	xP(x)
0	$\frac{1}{8}$	0	$0\left(\frac{1}{8}\right)$
1	$\frac{3}{8}$	2	$2\left(\frac{3}{8}\right)$
2	$\frac{3}{8}$	4	$4\left(\frac{3}{8}\right)$
3	$\frac{1}{8}$	6	$6\left(\frac{1}{8}\right)$

$$\text{Expected Value } \mu = \sum xP(x) = 0\left(\frac{1}{8}\right)+2\left(\frac{3}{8}\right)+4\left(\frac{3}{8}\right)+6\left(\frac{1}{8}\right)=\frac{24}{8}=3$$

The Expected Value is €3 i.e. the carnival owner must charge more than €3 to make a profit.
If he charges €3 euro the average profit per player is €0, and the game is fair.
However he must make a profit so he will charge more than the Expected Value.
If he charges €3.50 per game the average profit per game is 50 cent.
While individual players may make a profit, or lose or win back some of their costs, in the long run the carnival owner will win.

Example 3

You and a friend are playing the following game:

Two dice are rolled. If the total showing is a prime number, you pay your friend €6, otherwise, your friend pays you €2.

(i) What is the expected value of the game to you?

(ii) If you played the game 40 times, what are your expected winnings?

After playing the game for a while, you begin to think the rules are not fair and you decide to change the game.

(iii) How much (instead of €6) should you pay your friend when you lose so that your expected winnings are exactly €0?

Solution

	1	2	3	4	5	6
1	**2**	**3**	4	**5**	6	**7**
2	**3**	4	**5**	6	**7**	8
3	4	**5**	6	**7**	8	9
4	**5**	6	**7**	8	9	10
5	6	**7**	8	9	10	**11**
6	**7**	8	9	10	**11**	12

Outcome	Probability of each outcome, P(x)	Value associated with each outcome(€), x	xP(x)
Prime	$\frac{15}{36}$	–€6	$-6\left(\frac{15}{36}\right)=-\frac{5}{2}$
Non Prime	$\frac{21}{36}$	+€2	$2\left(\frac{21}{36}\right)=\frac{7}{6}$

(i) Expected Value $\mu = \sum xP(x) = -\frac{5}{2} + \frac{7}{6} = -\frac{4}{3}$

(ii) Expected winnings after 40 games $= 40\left(-\frac{4}{3}\right) = -€53.33 \Rightarrow$ Loss of €53.33

(iii) $\frac{15}{36}(x) + \frac{21}{36}(2) = 0$

$15x + 42 = 0$

$x = -2.8$

You should pay your friend €2.80

Example 4

For a particular age group, statistics show that the probability of dying in any one year is 1 in 1000 people and the probability of suffering some sort of disability is 3 in 1000 people. The Hope Life Insurance Company offers to pay out €20, 000 if you die and €10, 000 if you are disabled.
What profit is the insurance company making per customer based on the expected value if it charges a premium of €100 to its customers for the above policy?

Solution

Outcome	Value of each outcome, x	Probability of each outcome, P(x)	xP(x)
Dying	20,000	$\frac{1}{1000}$	20
Disability	10,000	$\frac{3}{1000}$	30
Neither of the above	0	$\frac{996}{1000}$	0
Expected Value			50

Charging €100 per customer, the company is expecting a profit of €50 per customer.