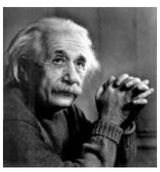
APPLICATIONS OF GEOMETRIC SEQUENCES AND SERIES TO FINANCIAL MATHS

"The most powerful force in the world is compound interest"
(Albert Einstein)



Loans and investments - associated terminology

People advertising loans and investment products want to make their products seem as attractive as possible. They often have different ways of calculating the interest, and the products might involve different periods of time. This makes it difficult for consumers to compare the products. Because of this, governments have rules about what information must be provided in advertisements for financial products and in the agreements that businesses make with their customers. Before defining terms such as APR used for loans, for which there is a statutory regulation, we need to look at the concept of present value.

Present Value

If you received €100 today and deposited it into a savings account, it would grow over time to be worth more than €100. This is a result of what is called the "time value of money", a concept which says that it is more valuable to receive €100 now rather than say a year from now. To put it another way the present value of receiving €100 one year from now is less than €100. Assuming a 10% interest rate per annum, the €100 I will receive in one years' time is worth $\frac{€100}{1.1} = €90.91$ now. That is its present value. (Investing €90.91 now for one year at 10% per annum yields €100 in one year's time.) The present value of €100 which I will receive in two year's time is $\frac{€100}{1.1^2} = €82.64$. (Investing €82.64 now for two year at 10% per annum yields €100 in two year's time.)

For example the owners of a piece of land might say that they will sell it to you now for €160,000 today or for €200,000 at the end of two years. Using a present value calculation you can see that the interest rate implicit in the second option is 11.8% per annum.

We will see that present value calculations can tell you such things as:

- The amount of each regular payment for a given loan (given the interest rate as an APR (see below) and the time in years over which the loan is to be repaid) (See example page 8)
- How much money to invest right now in return for specific cash amounts to be received in the future
- The size of the pension fund required on the date of retirement to give a fixed income every year for a certain number of years (See Sample paper 1 LCHL 2011 Q6)
- The fair market value of a bond (Pre Leaving NCCA paper 1 LCHL Q6
- The cash value option available in most US lottery games (LC HL 2011 Paper 1 Q6(d) and Q4 page 25 and Q6 Page 41)

Present value versus future value

When regular payments are being used to pay off a loan, then we are usually interested in calculating their ***present values*** (value right now), because this is the basis upon which the loan repayments and/or the APR are calculated. When regular payments are being used for investment, we may instead be interested in their ***future values*** (value at some time in the future), since this tells us how much we can expect to have when the investment matures.

Loans and other forms of credit – APR

In the case of loans and other forms of credit, there is a legal obligation to display the Annual Percentage Rate (APR) prominently. There are also clear rules laid down in legislation (Consumer Credit Act) about how this APR is to be calculated. Note that you need to be careful if you search the web or other resources for information about APR. The term is not used in the same way in all countries. We are concerned here only with the meaning of the term in Ireland, where its use is governed by Irish and European law.
(Formulae and Tables booklet page 31- statutory formula for APR)

APR is based on the idea of the **present value** of a future payment.

There are three key features of APR:

1. All the money that the customer has to pay must be included in the calculation – the loan repayments themselves, along with any set-up charges, additional unavoidable fees, etc.

2. The definition states that the APR is the annual interest rate (expressed as a percentage to at least one decimal place) that makes the present value of all of these repayments equal to the present value of the loan.

3. In calculating these present values, time must be measured in years from the date the loan is drawn down.

Note that the effect of this method of calculation is that the interest rate has the same effect as if a fixed amount of money was borrowed at this rate of annual interest, compounded annually.

APR takes account of the possible different compounding periods in different products and equalises them all to the equivalent rate compounded annually.

Nominal rate

If a credit interest rate is not an APR, then it may be referred to as a "**nominal rate**" or "headline rate". These have somewhat less relevance now, as it is no longer legal to quote an interest rate other than the APR in an advertisement for a loan or credit agreement.
Nonetheless, here is an example: if a loan or overdraft facility is governed by a charge of 1% per month calculated on the outstanding balance for that month, that might have been considered to be "nominally" a 12% annual rate, calculated monthly. However, it is actually an APR of 12.68%, since €1 owed at the start of a year would become $(1.01)^{12}$ = €1.1268 by the end of the year.
(See Page 32 Tables and Formulae Booklet.)

Savings and Investments - AER, EAR, CAR

For no particular reason, the term APR is reserved for loans and credit agreements, where the customer is borrowing from the service provider.
In the opposite case, where the customer is saving or investing money, the comparable term is the **Equivalent Annual Rate (EAR),** sometimes referred to as **Annual Equivalent Rate (AER)** or **Compound Annual Return / Compound Annual Rate (CAR)**.
The Financial Regulator's office considers these terms (EAR/AER/CAR) to be equivalent. The term CAR is approved for use in relation to tracker bonds. For other investment products, the Financial Regulator's office considers that the terms AER and EAR should be used.

The regulator's code itself uses the term "Equivalent Annual Rate", implying the acronym EAR, but AER may be more common internationally.

The rules governing their use in advertisements and agreements are not as clearly specified in law as is the case with the term APR, and it is not as clear what, if any, fees and charges have to be taken account of when calculating EAR/AER. Also, in the case of investments that do not have a guaranteed return, the calculation of EAR often involves estimates of future growth. Despite all this, the method of calculation is the exact same as is the case with the APR.

Example 1

Bank of Ireland offered a 9 month fixed term reward account paying 2.55% on maturity, for new funds from €10,000 to €500,000. (That is, you got your money back in 9 months, along with 2.55% interest.) Confirm that this was an EAR of 3.41%.

Solution

For every euro you ivested, you got back €1.0255 in $\frac{3}{4}$ of a year's time. At 3.41%, the present value of this return is $\frac{€1.0255}{1.0341^{0.75}} = €1$ which is as it should be.

Alternatively, just confirm that $€1.0341^{\frac{3}{4}} = €1.0255$, or that $€1.0255^{\frac{4}{3}} = €1.0341$.

<u>In summary:</u>

The future value or final value of the investment of €1 after 9 months is €1.0255.

The future value or final value of the €1 investment after 1 year is €1.0341.

The present value of €1.0341 which due in 1 year's time using this rate is €1.
The present value of €1.0255 due in 9 month's time using this rate is €1.

Example 2

The government's National Solidarity Bond offers 50% gross return after 10 years. Calculate the EAR for the bond.

Solution

For every €1 you invest you get back €1.5 in 10 years time.

$$F = P(1+i)^t$$
$$1.5 = 1(1+i)^{10}$$

$$1+i = (1.5)^{\frac{1}{10}}$$
$$1+i = 1.041379...$$
$$i = 0.041379...$$

EAR = 4.14%

Annuities

An annuity is a form of investment involving a series of periodic equal contributions made ***by*** an individual to an account for a specified term. Interest may be compounded at the end or beginning of each period. The term annuity is also used for a series of regular payments made ***to*** an individual for a specified time, such as in the case of a pension. The word annuity comes from the word "annual" meaning yearly.

Pension funds involve making contributions ***to an annuity*** before retirement and receiving payments ***from an annuity*** after retirement.

Calculations can be made to find out

(i) What a certain contribution per period amounts to as a fund

(ii) What size of contribution needs to be made to create of fund of a specific amount

When receiving payments from an annuity the ***present value* of the annuity** is the lump sum that must be invested now in order to provide those regular payments over the term.

Examples of annuities:

- Monthly rent payments
- Regular deposits in a savings account
- Social welfare benefits
- Annual premiums for a life insurance policy
- Periodic payments to a retired person from a pension fund
- Dividend payments on stocks and shares
- Loan repayments

The **future value of an annuity** is the total value of the investment at the end of the specified term. This includes all payments deposited as well as the interest earned.

The following extract is taken from "Mathematics A Practical Odyssey", Johnson Mowry

*The **present value of an annuity** is the lump sum that can be deposited at the beginning of the annuity's term, at the same interest rate and with the same compounding period, that would yield the same amount as the annuity. This value can help the saver to understand his or her options; it refers to an alternative way of saving the same amount of money in the same time. It is called the present value because it refers to the single action the saver can take in the present (i.e. at the beginning of the annuity's term) that would have the same effect as would the annuity.)*

(*See **2011 LCHL Sample Paper 1 Q6*** for an example of annuities in practice.)

Amortisation and amortised loans

The process of accounting for a sum of money by making it equivalent to a series of payments over time, such as arises when paying off a debt over time is called **amortisation**. Accordingly, a loan that involves paying back a fixed amount at regular intervals over a fixed period of time is called an **amortised loan**. Term loans and annuity mortgages (as opposed to endowment mortgages) are examples of amortised loans. See ***example 3*** below.

Bonds

A bond is a certificate issued by a government or a public company promising to repay borrowed money at a fixed rate of interest at a specified time. (See Q6 NCCA Pre Leaving 2011)

Regular payments over time – geometric series

Arrangements involving savings and loans often involve making a regular payment at fixed intervals of time. For example, a "regular saver" account might involve saving a certain amount of money every month for a number of years. A *term loan* or a *mortgage* might involve borrowing a certain amount of money and repaying it in equal instalments over time.
Calculations involving such regular payment schedules, when they are considered in terms of the *present values* of the payments as in loans - ***example 3*** below, (or the *future values* as in investments - ***example 4*** below) will involve the summation of a geometric series.

Amortised loan example

When regular payments are being used to pay off a loan, then we are usually interested in calculating their ***present values*** (value right now) rather than their future values, because this is the basis upon which the loan repayments and/or the APR are calculated.

We have seen that the APR is the interest rate for which the present value of all the repayments is equal to the present value of the loan. In the case of an amortised loan, these present values form a consistent pattern that turns out to be a geometric series.

Example 3

Seán borrows €10,000 at an APR of 6%. He wants to repay it in five equal instalments over five years, with the first repayment one year after he takes out the loan. How much should each repayment be?

Solution

Let each repayment equal A. Then the present value of the first repayment is $A/1.06$, the present value of the second repayment is $A/1.06^2$, and so on. The total of the present values of all the repayments is equal to the loan amount.

$$\text{Total of the present values of all the repayments} = \frac{A}{1.06} + \frac{A}{1.06^2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{A}{1.06^5}$$

This is a geometric series, with $n = 5$, first term $a = \dfrac{A}{1.06}$ and common ratio $r = \dfrac{1}{1.06}$.

The sum of the first 5 terms which is the loan amount is $S_5 = \dfrac{\dfrac{A}{1.06}\left(1 - \dfrac{1}{1.06^5}\right)}{\left(1 - \dfrac{1}{1.06}\right)} = 4.212363786A$

If S_5 has to equal the loan amount of €10,000, then $A = \dfrac{10000}{4.212363786} = €2373.96$ (i)

(Students could find S_n for a small number of terms by adding the terms individually first and then checking their answer by using the formula for S_n of a geometric series.)

This type of calculation is so common that it is convenient to derive a formula to shortcut the calculation for the regular repayment A. By considering the general case of an amortised loan with interest rate i, taken out over t years, for a loan amount of P, a geometric series can be used to derive the general formula: $A = P\dfrac{i(1+i)^t}{(1+i)^t - 1}$

This formula gives the same result as (i) above: $A = 10000\dfrac{0.06(1.06)^5}{\left((1.06)^5 - 1\right)} = €2373.96$

(The formula assumes payment at the end of each payment period.
We will derive this formula later. **(**See page 14)

Amortised loans - regular payments at intervals other than yearly

The calculations involved are the same, except that one must be careful to treat the APR properly. Suppose, in the above case, Seán wanted to make monthly repayments instead of yearly ones. How much would each repayment then be? There are two ways to approach this.
We can either keep the year as the unit of time, or change to months.
If we stay with years, we keep the same rate for the APR but then we need to deal with fractional units of time. If we switch to months, we must convert the APR to an equivalent monthly compounded rate and then we have integer amounts of time.

1. Stick with years

If we keep the year as the unit of time, then the first repayment is made after $\frac{1}{12}$ of a year, so its present value is $\frac{A}{1.06^{\frac{1}{12}}}$ the present value of the second repayment is $\frac{A}{1.06^{\frac{2}{12}}}$, and so on.
In the case of example 3 on the previous page, if Seán switches to monthly payments, the geometric series will have 60 terms, with $a = \frac{A}{1.06^{\frac{1}{12}}}$, and $r = \frac{A}{1.06^{\frac{1}{12}}}$.

The sum of the present values of all the repayments = Loan amount.

$$S_{60} = \frac{\frac{A}{1.06^{\frac{1}{12}}}\left(1-\left(\frac{1}{1.06^{\frac{1}{12}}}\right)^{60}\right)}{\left(1-\frac{1}{1.06^{\frac{1}{12}}}\right)} = 51.92382159A$$

51.923A = Loan amount = €10,000, giving A = €192.59.

(Note: If we multiply this answer by 12, we get €2311.08 which is less than the amount Sean repaid when repaying yearly. How much will this save him over the five years?)

2. Switch to months

If we switch to months as the unit of time, then we first have to determine what monthly interest rate, compounded monthly, is equivalent to an APR of 6%. This involves getting the twelfth root of 1.06, which is 1.004867551. Then, we can treat the present values as a geometric series again, with first term $a = \frac{A}{1.004867551}$, common ratio $r = \frac{1}{1.004867551}$ and number of terms n= 60.

$$S_{60} = \frac{\frac{A}{1.004867551}\left(1-\left(\frac{1}{1.004867551}\right)^{60}\right)}{\left(1-\frac{1}{1.004867551}\right)} = 51.923A$$

As above this gives A = €192.59.

Alternatively, we can use the amortisation formula with
i = 0.004867551, P = 10,000, and t = 60 (using months as the time periods and the corresponding i).

$$A = 10000\frac{0.004867551(1.004867551)^{60}}{(1.004867551)^{60} - 1} = €192.5898329 = €192.59$$

Regular saver accounts and similar investments

When regular payments are being used for savings or investment, we are interested in their ***future values*** (value at some time in the future), since this tells us how much we can expect to have when the investment matures.

As mentioned earlier, any regular payment over time will give rise to a geometric series, irrespective of whether its purpose is to repay a loan or to generate savings for the future. Accordingly, the analysis is very similar. In the case of savings and investments, we are generally interested in the future value rather than the present value.

Example 4

According to "itsyourmoney.ie", one of the banks is offering a regular monthly savings account with an AER of 4.00% on balances up to €15,000. If I save €100 per month, starting today, how much will I have in the account in five years' time, assuming the rate stays the same?

Solution:

(Using years as the unit of time)

The future value (value in 5 years' time) of my first €100 is $100(1.04)^5$.

The future value (value in 5 years' time) of my second €100 is $100(1.04)^{(59/12)}$.

The future value (value in 5 years' time) of my 60th (and last) €100 is $100(1.04)^{(1/12)}$.

The future value of all of these regular payments into the savings account is:

$$100(1.04)^5 + 100(1.04)^{\frac{59}{12}} + 100(1.04)^{\frac{58}{12}} + \ldots 100(1.04)^{\frac{2}{12}} + 100(1.04)^{\frac{1}{12}}$$

This is a geometric series with first term $100(1.04)^5$, common ratio $(1.04)^{(-1/12)}$ and 60 terms.

$$S_{60} = \frac{100(1.04)^5\left(1 - \left(\frac{1}{1.04^{\frac{1}{12}}}\right)^{60}\right)}{\left(1 - \frac{1}{1.04^{\frac{1}{12}}}\right)} = €6639.57$$

(To make calculations easier, store $\frac{1}{1.04^{\frac{1}{12}}}$ in the calculator's memory and just recall it each time it is needed.)

Another option, to avoid fractions, when adding up the terms of the geometric series is to reverse the order: $S_n = 100(1.04)^{\frac{1}{12}} + 100(1.04)^{\frac{2}{12}} + \ldots + 100(1.04)^{\frac{58}{12}} + 100(1.04)^{\frac{59}{12}} + 100(1.04)^5$

$$a = 100(1.04)^{\frac{1}{12}}, r = 1.04^{\frac{1}{12}}, n = 60$$

$$S_{60} = \frac{100(1.04)^{\frac{1}{12}}\left(\left((1.04)^{\frac{1}{12}}\right)^{60} - 1\right)}{(1.04)^{\frac{1}{12}} - 1} = €6,639.57$$

Connection between the calculations involved in an amortised loan and those involved in a regular saver account

There is clearly a connection between the calculations involved in an amortised loan and those involved in a regular saver account. It may be noted that we have a formula on page 31 of the "formulae and tables" booklet for calculating the repayments involved in an amortised loan but we don't have a formula for the equivalent problem of calculating the annual or monthly savings required to generate a particular investment fund in the future.

However, if we wish to take advantage of the loan formula to shortcut the consideration of the geometric series involved in the savings case, then we can readily do so by observing that a single jump can take us from the present value to the future value of any fund, or back again.
That is, any given stream of payments has a total present value and a total future value. We have already seen how we can jump forward using the compound interest formula or backwards using present value formula (first two formulae on page 30 of the "formulae and tables" book).

*Thus, for example, if we want to know what monthly investment (paid at the end of each month) is required to generate a fund of €20,000 in 10 years time at an EAR of 5%, we can instead solve the equivalent problem of finding the monthly repayments required to pay back a loan whose principal is the present value of €20,000, and which is to be paid off over 10 years at an APR of 5%. Thus, we can first find the present value of €20,000, and then use the amortised loan repayment formula to find the monthly investment required to generate the €20,000 in 10 years time.
The present value of € 20,000, invested at an EAR of 5% for 10 years is:

$$P = \frac{F}{(1+i)^t} = \frac{20000}{(1.05)^{10}} = €12278.26507$$

Using the amortised loan formula to calculate the monthly investment required to generate a fund of €20,000 where the monthly interest rate is $(1.05)^{1/12}$ = 1.00474124:

$$A = 12278.26507\frac{0.0040741024(1.004074124)^{120}}{(1.004074124)^{120} - 1} = €129.56.$$

Hence a monthly investment of €129.56 would generate a fund of €20,000 in 10 years time at an EAR of 5%.

(See Q6 LCHL Sample paper 1 2011 - Padraig needs €358,710.84 in 40 years time in a retirement fund - what fixed amount should he invest every month at an EAR of 3% to achieve this fund?)

Use of ratio and proportion

Another possible approach to determining the monthly investment required to generate a particular fund is to use ratio/proportion. One can first calculate, using a geometric series, the fund that would be generated by a monthly saving of €1 (or €100). By comparing this amount to the fund required, we can scale this saving proportionally upwards or downwards as required, since the final fund is in direct proportion to the monthly amount saved.

€1 invested at the end of each month for 120 months at an EAR of 5% would give the following geometric series whose sum is:

$$\begin{aligned} S_n &= 1(1.004074124)^{119} + 1(1.004074124)^{118} + \ldots\ldots\ldots\ldots 1(1.004074124)^{1} + 1 \\ &= 1 + 1(1.004074124)^{1} + \ldots\ldots\ldots + 1(1.004074124^{118} + 1(1.004074124^{119} \\ &= 1\frac{\left(1.004074124^{120} - 1\right)}{1.004074124 - 1} \\ &= 154.3631531 \end{aligned}$$

Hence if one invests €1 at the end of each month for 10 years at an EAR of 5% the fund generated is €154.3631531.
To generate a fund of €20,000 one would need to invest €20,000/ €154.3631531= €129.56

Amortisation formula (Page 31 "Formulae and Tables")

Terms associated with the amortisation formula revisited:

Present Value is the value on a given date of a future payment or series of future payments discounted to reflect the time value of money and other factors such as investment risk.

An **annuity** is a series of equal payments or receipts that occur at evenly spaced intervals. Each payment occurs at the end of each period for an **ordinary annuity**.

An **amortised loan** is a loan for which the loan amount plus interest is paid off in a series of regular payments. An amortised loan is an annuity whose future value is the same as the loan amount's future value, under compound interest. An amortised loan's payments are used to pay off a loan. Other types of annuities' payments can be used to generate savings as for example for retirement funds.

We can think of the situation in two ways which give the same end result:

1) The sum of the present values of all the annual repayment amounts = sum borrowed.

(This principle is enshrined in European Law)

2) Future value of loan amount = Future value of the annual repayment amounts (i.e. future value of the annuity)

Note: Compounding periods other than annual can be applied e.g. quarterly, monthly, daily etc.

Given that A = annual repayment amount, the present value of one annual repayment amount paid in t years time is

$P = \frac{A}{(1+i)^t}$, where i is the annual rate of interest expressed as a decimal or fraction

So if I borrow €10,000 over 5 years, when I add up the present values of all the annual repayment amounts, this sum should equal €10,000.

$$10,000 = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \frac{A}{(1+i)^4} + \frac{A}{(1+i)^5} = A\left(\frac{1}{(1+i)^1} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4} + \frac{1}{(1+i)^5}\right)$$

Students see that the present value of each repayment is less than the one before it.

Two methods of deriving the "**Amortisation – mortgages and loans**" formula (Page 31 Formulae and Tables book) are used below.

Method 1

Loan amount = sum of the present value of all the repayments

(assuming payment at the end of each payment period)

P = Loan amount , A = periodic repayment amount, t = the number of payment periods

i = the interest rate for the payment period expressed as a decimal or fraction

$$P = \frac{A}{(1+i)^1} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + \ldots\ldots\ldots\ldots\ldots + \frac{A}{(1+i)^t}$$

$P = S_n$ of a geometric series , $n = t$ = number of compounding periods , $a = \frac{A}{1+i}$, $r = {}^{1}\!/_{1+i}$

$$P = \frac{a(1-r^n)}{1-r} = \frac{\left(A/(1+i)\right)\left(1-\left(1/(1+i)\right)^t\right)}{\left(1-1/(1+i)\right)}$$

$$= \frac{\left(A/(1+i)\right)\left(\frac{1(1+i)^t-1}{(1+i)^t}\right)}{\left(\frac{1+i-1}{1+i}\right)}$$

$$= \frac{\left(A/(1+i)\right)\left(1(1+i)^t-1\right)}{\left(\frac{i}{1+i}\right)(1+i)^t}$$

$$P = \frac{(A)\left((1+i)^t-1\right)}{i(1+i)^t}$$

$$\Rightarrow A = P\frac{i(1+i)^t}{(1+i)^t-1}$$

Method 2

The future value of the loan amount P = sum of the future values of t equal repayments each of value A made at the end of each compounding period.

$$P(1+i)^t = A(1+i)^{t-1} + A(1+i)^{t-2} + \ldots\ldots\ldots\ldots + A(1+i)^2 + A(1+i) + A$$

$$= A + A(1+i) + A(1+i)^2 + A(1+i)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots A(1+i)^{t-1}$$

$P(1+i)^t = S_n$ of a geometric series

$$S_n = \frac{a(r^n-1)}{r-1} \text{ where } a = A,\ r = 1+i, n = t$$

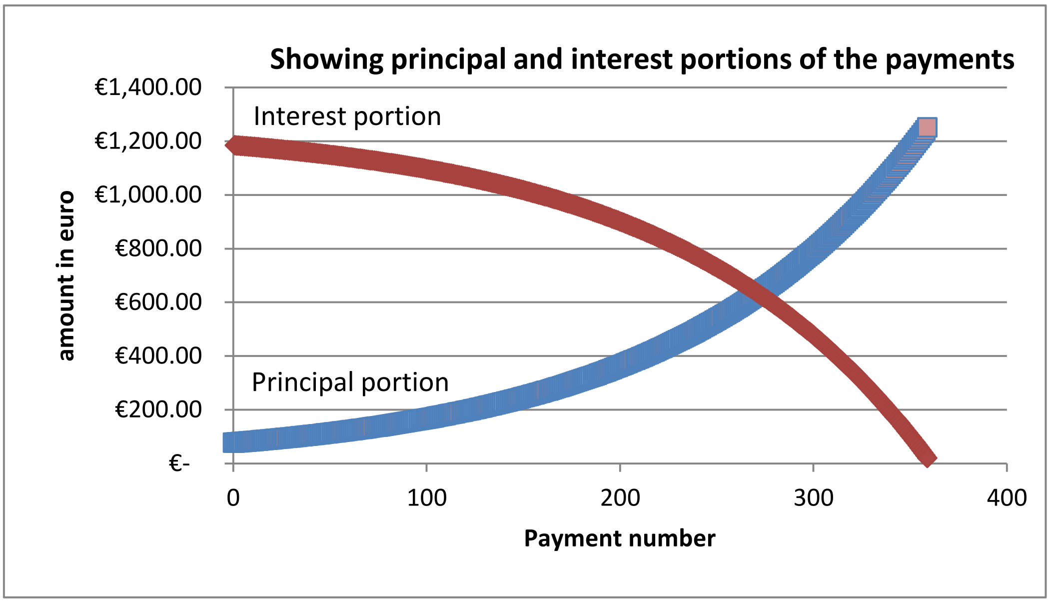
$$P(1+i)^t = \frac{A((1+i)^t-1)}{(1+i)-1}$$

$$P(1+i)^t = \frac{A((1+i)^t-1)}{i}$$

$$\Rightarrow A = P\frac{(1+i)^t i}{(1+i)^t-1}$$

Exercise: Adapt the formula for the situation where the repayments are made **at the beginning of** each payment period as opposed to at the end of each payment period.

Assuming a loan is repaid in fixed annual repayments – the annual repayment is made up of two parts – one part interest and the remainder is part of the capital sum borrowed, called "principal portion" in the graph below. The graphs below show that that even though the periodic payment is fixed, the part of it which is interest is decreasing as more of the loan is paid off and the part of it which is principal is increasing. The graph below refers to an amortisation schedule for a loan paid back monthly over 360 months. (See Sample questions Set A question 8.)



Amortisation Schedule

An amortisation schedule is a list of several periods of payments, the principal and interest portions of those payments and the outstanding principal (or balance) after each of those payments is made.

Below is the amortisation schedule for example 3 Page 8, showing in figures the trends in the interest and principal portions of each payment for successive payments for the loan in example 3:

€10,000 loan paid back over 5 years at 6% APR involving a fixed annual repayment of €2376.96 per year.

Payment #	Fixed payment	Interest portion	Principal portion	balance		
0				€ 10,000.00	**rate**	6.00%
1	€ 2,373.96	€ 600.00	€ 1,773.96	€ 8,226.04	**years**	5
2	€ 2,373.96	€ 493.56	€ 1,880.40	€ 6,345.63	**fixed payment per year**	€ 2,373.96
3	€ 2,373.96	€ 380.74	€ 1,993.23	€ 4,352.41		
4	€ 2,373.96	€ 261.14	€ 2,112.82	€ 2,239.59		
5	€ 2,373.96	€ 134.38	€ 2,239.59	€ 0.00		

See Appendix 1 for the formulae for this spreadsheet.

Steps in an amortisation schedule:

1. Fill in the first balance = loan amount (payment number 0)
2. For payment 1, fill in the payment number and fixed repayment amount
3. For payment 1 row, find the interest on the previous balance using the simple interest formula
4. Calculate the debt payment (principal portion) = the repayment amount - the interest portion
5. Calculate the new balance = previous balance - the principal portion
6. Repeat Steps 3, 4 and 5 for all the other payments from payment 2 onwards
7. For the last payment, the principal portion = the previous balance
8. When all the payments have been made the final balance is €0.00

Explanation of the amortisation schedule

Payment #	total payment	Interest portion	Principal portion	Balance
0				Loan amount
1	Fixed repayment amount calculated using "Amortisation - loans and mortgages formula"	Simple interest on the previous balance	Repayment amount - interest portion	Previous balance - this payment's principal portion
2	Fixed repayment amount calculated using "Amortisation - loans and mortgages formula"	Simple interest on the previous balance	Repayment amount - interest portion	Previous balance - this payment's principal portion
Last	Fixed Repayment = principal portion + interest portion	Simple interest on the previous balance	Previous balance	€ 0.0

Using JC knowledge of compound interest to show the outstanding balance at the end of each payment period, and the interest and principal portion of each payment for example 3 page 8:

Paying back a loan of €10,000 over 5 years at an APR of 6% in fixed annual repayments of €2,373.96

A	B	C	D	E	F	G
Fixed Repayment A per year	€2,373.96	interest portion	Principal portion	Present Value		
Starting Balance	€10,000.00					
interest for year 1	€600.00	€600.00	€1,773.96	2373.96/(1.06)^5=	€1,773.96	
Amount owing at end of 1 year	€10,600.00					
Payment 1	€2,373.96					
Balance at end of year 1	€8,226.04					
interest for year 2	€493.56	€493.56	€1,880.40	2373.96/(1.06)^4=	€1,880.40	
Amount owing at end of year 2	€8,719.60					
Payment 2	2373.964					
Balance at end of year 2	€6,345.63					
interest for year 3	€380.74	€380.74	€1,993.22	2373.96/(1.06)^3=	€1,993.23	
Amount owing at end of 3 years	€6,726.37					
Payment 3	2373.964					
Balance at end of year 3	€4,352.41					
interest for year 4	€261.14	€261.14	€2,112.82	2373.96/(1.06)^2=	€2,112.82	
Amount owing at end of 4 years	€4,613.55					
Payment 4	2373.964					
Balance at end of year 4	€2,239.59					
interest for year 5	€134.38	€134.38	€2,239.58	2373.96/(1.06)^1=	€2,239.59	
Amount owing at end of 5 years	€2,373.96					
Payment 5	2373.964					
Balance at end of year 5	€0.00					
Sum of principal portions of the repayments = sum of the present values of all the repayments = loan amount			**€10,000.00**		**€10,000.00**	

A mortgage of €P is taken out and is to be repaid over t years, with equal payments of €A being made at the end of each year.
The APR expressed in decimal form is i.

(i) Show that the sum of the present values of all the repayments is equal to the loan amount

(ii) Derive an expression for the payment amount A, in terms of P, i, and t.

t/years	A/€	Balance outstanding
0		P
1	A	$P(1+i)-A$
2	A	$P(1+i)^2-A(1+i)-A$
3	A	$P(1+i)^3-A(1+i)^2-A(1+i)-A$
4	A	$P(1+i)^4-A(1+i)^3-A(1+i)^2-A(1+i)-A$
t	A	$P(1+i)^t-A(1+i)^{t-2}-A(1+i)^{t-1}-A(1+i)-A$

What is the balance outstanding after t years equal to?

Since the final payment has now been made and the loan is paid off, the balance outstanding at the end of year t is zero.

$$P(1+i)^t-A(1+i)^{t-1}-A(1+i)^{t-2}-\cdots-A(1+i)-A=0$$

$$\Rightarrow P(1+i)^t=A(1+i)^{t-1}+A(1+i)^{t-2}+\cdots+A(1+i)+A$$

Dividing both sides of the equation by $(1+i)^t$:

$$P=\frac{A}{(1+i)}+\frac{A}{(1+i)^2}+\frac{A}{(1+i)^3}+\cdots+\frac{A}{(1+i)^{t-1}}+\frac{A}{(1+i)^t}$$

The loan amount = the sum of the present values of all the repayments

Question 6 (50 marks)
Pádraig is 25 years old and is planning for his pension. He intends to retire in forty years' time, when he is 65. First, he calculates how much he wants to have in his pension fund when he retires. Then, he calculates how much he needs to invest in order to achieve this. He assumes that, in the long run, money can be invested at an inflation-adjusted annual rate of 3%. Your answers throughout this question should therefore be based on a 3% annual growth rate.

(a) Write down the present value of a future payment of €20,000 in one years' time.

(b) Write down, in terms of t, the present value of a future payment of €20,000 in t years' time.

(c) Pádraig wants to have a fund that could, from the date of his retirement, give him a payment of €20,000 at the start of each year for 25 years. Show how to use the sum of a geometric series to calculate the value on the date of retirement of the fund required.

Pádraig plans to invest a fixed amount of money every month in order to generate the fund calculated in part **(c)**. His retirement is $40 \times 12 = 480$ months away.

- **(i)** Find, correct to four significant figures, the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate of 3%.
- **(ii)** Write down, in terms of n and P, the value on the retirement date of a payment of €P made n months before the retirement date.
- **(iii)** If Pádraig makes 480 equal monthly payments of €P from now until his retirement, what value of P will give the fund he requires?
- **(iv)** If Pádraig waits for ten years before starting his pension investments, how much will he then have to pay each month in order to generate the same pension fund?

(a) Write down the present value of a future payment of €20,000 in one years' time.

A: P = 20000/1.03= €19,417.48

(Note: €19,417.48 would increase to €20000 in one year at a 3% annual growth rate.)

(b) Write down in terms of t, the present value of a future payment of €20000 in t years' time.

A: $P=20000/(1.03)^t$

(c) Padraig wants to have a fund that could, from the date of his retirement, give him a payment of €20,000 at the start of each year for 25 years. Show how to use the sum of a geometric series to calculate the value on the date of retirement of the fund required.

A: The amount of money in the fund on the date of retirement = sum of the present values of all the payments up to the date of retirement.

Present value of the first payment $= 20000$

Present value of the second payment $= \dfrac{20000}{1.03}$

Present value of the third payment $= \dfrac{20000}{(1.03)^2}$

Present value of the first payment $= \dfrac{20000}{(1.03)^{24}}$

Fund generated on the date of retirement $= 20000 + \dfrac{20000}{1.03} + \dfrac{20000}{(1.03)^2} + \ldots\ldots\ldots + \dfrac{20000}{(1.03)^{24}}$

Fund $= S_n$ of a geometric series

$$S_n = \frac{a(1-r^n)}{1-r} \text{ where } a = 20000, r = \frac{1}{1.03}, n = 25$$

$$S_n = \frac{20000(1-\left(\frac{1}{1.03}\right)^{25})}{\left(1-\frac{1}{1.03}\right)} = €358,710.84$$

Payment number	Present value
	€20,000.00
1	€19,417.48
2	€18,851.92
3	€18,302.83
4	€17,769.74
5	€17,252.18
6	€16,749.69
7	€16,261.83
8	€15,788.18
9	€15,328.33
10	€14,881.88
11	€14,448.43
12	€14,027.60
13	€13,619.03
14	€13,222.36
15	€12,837.24
16	€12,463.34
17	€12,100.33
18	€11,747.89
19	€11,405.72
20	€11,073.52
21	€10,750.99
22	€10,437.85
23	€10,133.83
24	€9,838.67
Sum of present values	€358,710.84

Students note that this is less than 20000*(25).

6(d)(i)

Pádraig plans to invest a fixed amount of money every month in order to generate the fund calculated in part(c). His retirement is 40 × 12 = 480 months away months away. Find correct to four significant figures the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate of 3%.

Suggested solution:

$(1+i)^{12} = 1.03$

$(1+i) = 1.03^{1/12} = 1.002466$

i= rate of interest per month = 0.002466 = 0.2466%

6(d)(ii)

Write down in terms of n and P, the value on the retirement date of a payment of €P made n months before retirement date.

Suggested solution:

The future (final) value of a payment of €P paid n months before retirement date is

$P(1+ 0.002466)^n$

6(d)(iii)

If Pádraig makes 480 equal monthly payments of €P **from now** until his retirement, what value of P will give the fund he requires?

Suggested solution:

The future value of the annuity = €358710.84, $i = 0.002466$, monthly payment = €P

$$358710.84 = P(1+i)^{480} + P(1+i)^{479} + \ldots\ldots + P(1+i)^2 + P(1+i)^1$$

$$= P(1+i)^1 + P(1+i)^2 + \ldots\ldots + P(1+i)^{479} + P(1+i)^{480} \text{ (in reverse order)}$$

Right hand side is S_n of a geometric series

$$S_n = \frac{a(r^n - 1)}{r-1} \text{ where } a = P(1+i), r = (1+i), n = 480$$

$$358710.84 = \frac{P(1.002466)(1.002466^{480} - 1)}{1.002466 - 1} = P(919.38)$$

$$P = \frac{358710.84}{919.38} = €390.17$$

6(e)

If Pádraig waits for ten years before starting his pension investments, how much will he then have to pay each month in order to generate the same pension fund?

Suggested solution:

Future value of the annuity = €358,710.84

10 years = 120 months $\Rightarrow$ retirement is 360 months away

$$358710.84 = P\left[\frac{1.002466\left((1.002466)^{360}-1\right)}{1.002466-1}\right] = P\left[\frac{1.002466(2.427027332-1)}{0.002466}\right] = P\left[\frac{1.002466(1.427027332)}{0.002466}\right]$$

$$= P(580.11)$$

$$P = \frac{358710.84}{580.11} = €618.35$$

(a) A graduate is setting up his own computer company. He borrows the €5000 for set-up costs for 6 months at a flat rate of 1% per month (compounded monthly).
He wants to arrange to pay this off in equal monthly instalments.

(i) Calculate the monthly repayment amount.

(ii) Make a schedule showing the monthly payment, the monthly interest on the outstanding balance, the portion of the payment contributing toward reducing the debt, and the outstanding balance.

Suggested solution:

(i)

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1}$$

$$A = 5000\frac{(0.01)(1.01)^6}{(1.01)^6 - 1} = \frac{53.07600753}{0.06152015} = €862.74$$

(ii)

Payment #	Payment	Interest	Debt Payment	Balance
0				€5000.00
1	€862.74	€50	€812.74	€4187.26
2	€862.74	€41.87	€820.87	€3366.39
3	€862.74	€33.66	€829.09	€2537.31
4	€862.74	€25.37	€837.37	€1699.94
5	€862.74	€17.00	€845.74	€854.20
6	€862.74	€8.54	€854.20	€0.00

(b) After 5 years the company needs to raise money to expand. It proposes to issue a 10-year €2,000 bond that will pay €100 at the beginning of every year. If the current market interest rate is 5% per annum, what is the fair market value of this bond? Explain your answer and justify any assumptions you make.

(Bond: A bond is a certificate issued by a government or a public company promising to repay borrowed money at a fixed rate of interest at a specified time.)

Suggested Solution:

This bond promises two things. The bond holder gets €2000 when the bond matures in 10 years time. In the meantime they get €100 per year at the beginning of each year. The question being asked is what would be a reasonable price for the company selling the bond to charge the person buying the bond.

The present value of €2000 due in 10 years' time is calculated as follows:

Present Value (P) of €2000 due in 10 years = $\dfrac{2000}{(1.05)^{10}} = \dfrac{2000}{1.62889} = €1227.83$

The present value of ten payments of €100 per year for ten years:

In order to be able to pay the bond holder €100 at the beginning of the first year, the company will now need to charge them €100.

In order to be able to pay them €100 in 1 year's time (at the beginning of the second year) the company will need to charge them the present value of €100 due in 1 year calculated as follows:

$$P=\frac{100}{(1+0.05)^1}=\frac{100}{1.05}=€95.24.$$

In order to be able to pay them €100 in 2 years' time (at the beginning of the third year) the company will need to charge them the present value of €100 due in 2 years' time calculated as follows:

$$P_1=\frac{100}{(1+0.05)^2}=\frac{100}{1.1025}=€90.70.$$

The pattern continues in this way.

The total amount that the investor will need to pay the company in order to be able to get €100 at the beginning of each year for the next 10 years is the sum of the present values of all the payments of €100.

$$S_{10}=100+\frac{100}{1.05}+\frac{100}{(1.05)^2}+\frac{100}{(1.05)^3}+\frac{100}{(1.05)^4}+\frac{100}{(1.05)^5}+\frac{100}{(1.05)^6}+\frac{100}{(1.05)^7}+\frac{100}{(1.05)^8}+\frac{100}{(1.05)^9}$$

The right hand side of this equation is a Geometric series.

$a = 100$, $r = \dfrac{1}{1.05}$ and $n = 10$

$$S_{10}=100\left[\frac{1-\left(\frac{1}{1.05}\right)^{10}}{1-\frac{1}{1.05}}\right]=100\left[\frac{1-(0.95238)^{10}}{1-0.95238}\right]=100\left[\frac{1-0.6139}{0.04762}\right]=100[8.10793]=810.79$$

The company needs to charge the bondholder €810.79 now in order to be able to pay him/her €100 per year for 10 years.

Hence in order to be able to pay the investor €100 per year for the next 10 years and pay €2000 at the end of the 10 years, the fair market value of the bond is €1227.83 + €810.79 = €2038.62.

Note: This answer is based on the fact that the €100 is paid at the beginning of every year.
(Hence you get the first €100 immediately after the investment is made.)

1. **Regular savings (Future value of an annuity)**
Sample question: At the end of each month a deposit of €500 is made into an account that pays an AER of 8% compounded monthly. What will the final amount be after 5 years?

2. **Regular savings (Future value of an annuity)**
Sample question: Sonya deposits €300 at the end of each quarter in her savings account. If the money earns 5.75% (EAR), how much will this investment be worth at the end of 4 years?

3. **Regular savings (We sometimes hear the phrase "sinking fund" i.e. the future value has been decided and you "sink" regular amounts into the fund to reach this target. Find the value of the regular payment.)**
Sample question: Robert needs €5000 in three years. How much should he deposit at the end of each month in an account that pays 8% (EAR) in order to achieve his goal?

4. **Present value of an annuity with a finite number of terms**
Sample question: You have won a prize in a lottery. The prize entitles you to €1,000 per month, paid at the end of each month, for the next 20 years.
However, you prefer to have the entire amount now in a lump sum.
If the EAR is 8%, how much will you accept as a lump sum?

5. **Present value of an annuity with a finite number of terms**
Sample question: Jack won a prize in a lottery. He has been given a choice of two options:
Option A: Receive an annuity of €1500, ***at the beginning of*** each month for 25 years.
Option B: Take a lump sum instead.
Jack decides to take Option B.
a) What lump sum should he accept assuming an AER of 4%?
b) He invests the lump sum he receives, for 20 years, in an account that pays 4% (AER).
How much will Jack's investment have amounted to after the 20 years?

See also - Q6 2011 LCHL Mathematics Paper 1 www.examinations.ie

6. **Present value of an annuity with an infinite number of terms**
Sample question: Suppose that you expect to receive a payment of €100 once per year for an indefinitely long time. What is the present value of this annuity?
(An annuity is a sum of money to be paid in regular intervals.)

7. **Savings Bonds**
Sample question: *New Horizons Computer Company* needs to raise money to expand. It issues a 10-year bond of €1,000 that pays €30 at the end of every six month period for the 10 years. If the current market annual interest rate 7.12% (AER), with interest added at six-month intervals, what is the fair market value of the bond?

8. **Prepaying a loan**
Sample question: Mr. Mooney bought his house in 1975. He obtained a loan from the bank for 30 years at an interest rate of 9.8% APR. His monthly payment at the end of each month was €1260. In 1995, Mr. Mooney decided to pay off the loan. Find the balance of the loan he still owed at that time.

<u>Sample question 1 LCHL</u> Regular savings (Future value of an annuity)
At the end of each month a deposit of €500 is made into an account that pays an AER of 8% compounded monthly. What will the final amount (i.e. future value) be after 5 years?

Since this is a question on savings/investments it is a good idea to look at future values.

<u>Suggested solution:</u>

Future value of the first €500 = $500((1.08)^{59/12})$

Future value of the second €500 = $500((1.08)^{58/12})$

Future value of the third €500 = $500((1.08)^{57/12})$

Future value of the second last €500 = $500((1.08)^{1/12})$

Future value of the last €500 = €500 as it is invested at the end of the last payment period

The sum of all the future values of all the regular savings of €500 per month is

$$500 + 500(1.08)^{1/12} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 500(1.08)^{57/12} + 500(1.08)^{58/12} + 500(1.08)^{59/12}$$

The sum of these terms is Sn of a geometric series :

$$Sn = \frac{a(r^n - 1)}{r - 1} \text{ with } a = 500, r = (1.08)^{1/12} \text{ and } n = 60$$

$$Sn = \frac{500\left[\left((1.08)^{1/12}\right)^{60} - 1\right]}{1.08^{1/12} - 1} = \frac{234.6640384}{0.00643403} = €36,472.33$$

<u>Sample question 2 LCHL</u> Regular savings (Future value of an annuity)
Sonya deposits €300 at the end of each quarter in her savings account. If the money earns 5.75% (EAR), how much will the investment be worth at the end of 4 years?

Since this is a question on savings/investments it is a good idea to look at future values.

<u>Suggested Solution</u>

Future value of the first €300 = $300(1.0575)^{15/4}$

Future value of the second €300 = $300(1.0575)^{14/4}$

Future value of the third €300 = $300(1.0575)^{13/4}$

Future value of the second last €300 = $300(1.0575)^{12/4}$

Future value of the last €300 = €300

The sum of all the future values of all the regular savings of €300 per month is

$300 + 300(1.0575)^{1/4} + \dots\dots\dots\dots + 300(1.0575)^{13/4} + 300(1.0575)^{14/4} + 300(1.0575)^{15/4}$

This is a geometric series with $a = 300, r = (1.0575)^{1/4}$ and $n = 16$

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{300\left[\left((1.0575)^{1/4}\right)^{16} - 1\right]}{1.0575^{1/4} - 1} = \frac{75.18266063}{0.014075041} = €5,341.56$$

Sample question 3 LCHL

Regular savings ("Sinking fund" – the future value has been decided and you "sink" regular amounts into the fund to reach this target. Find the value of the regular payment.)

Robert needs €5000 in three years. How much should he deposit at the end of each month in an account that pays 8% (EAR) in order to achieve his goal?

Since this is a question on savings/investments it is a good idea to look at future values.

The future/final value (FV) of his investment needs to be €5000.

Assume that he makes a regular saving of €x per month.

€5000 = sum of the future values of all the regular payments of €x into the fund.

Future value of the first €$x = x(1.08)^{35/12}$

Future value of the second €$x = x(1.08)^{34/12}$

Future value of the third €$x = x(1.08)^{33/12}$

Future value of the second last €$x = x(1.08)^{1/12}$

Future value of the last €x = €x

The sum of all the future values of all the regular savings of €x per month is

$5000 = x + x(1.08)^{1/12} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; x(1.08)^{33/12} + x(1.08)^{34/12} + x(1.08)^{35/12}$.......equation (i)

This is a geometric series with $a = x, r = (1.08)^{1/12}$ and $n = 36$

$$5000 = \frac{a(r^n - 1)}{r - 1} = \frac{x((1.08)^{1/12})^{36} - 1)}{1.08^{1/12} - 1} = 40.36536907x$$

$$x = €123.87$$

Note:

$$(1.08)^{\frac{1}{12}} = 1.006434030$$

$$i = 0.006434030$$

Alternative view (Sample question 3):

We could also view this as the present value of €5000 = present value of all the monthly payments.
In other words find the value of each of the equal monthly repayments needed to pay back a loan equal to the present value of €5000.
(We can see this from equation (i) on the previous page if we divide both sides by $(1+i)^{36} = (1.08)^{36/12}$.)

$$5000 = x + x(1.08)^{1/12} + \ldots\ldots\ldots\ldots\ldots x(1.08)^{33/12} + x(1.08)^{34/12} + x(1.08)^{35/12} \ldots\text{equation(i)}$$

$$\frac{5000}{(1.08)^{36/12}} = \frac{x}{(1.08)^{36/12}} + \frac{x}{(1.08)^{35/12}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{x}{(1.08)^{1/12}}$$

PV of €5000 = Present value of last payment +...................................... + Present value of first payment

We can sum the right hand side using a geometric series and hence find x.
Alternatively we can use the "amortisation- mortagages and loans" formula to find the amount of the regular payment x, (A in the formula) where P in the formula is the present value of the €5000, and i is the monthly interest rate.

Using the amortisation formula (to find *A*)

Amortisation formula Page 31 Tables and Formulae: $A = P\dfrac{i(1+i)^t}{(1+i)^t - 1}$

First we find the present value of €5000 due in 3 years time assuming an EAR of 8%

$P = \dfrac{5000}{(1.08)^3} = €3969.16$ Substituting this for P into the amortisation formula we can find A, the amount which should be deposited each month.

$$(1.08)^{\frac{1}{12}} = 1.006434030$$
$$i = 0.006434030$$

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} = 3969.16\frac{(0.006434030)(1.08)^3}{((1.08)^3 - 1)} = €123.87$$

Sample question 4 LCHL
Present value of an annuity with a finite number of terms
You have won a prize in a lottery.
The prize entitles you to €1,000 per month at the end of each month for the next 20 years.
However, you prefer to have the entire amount now in a lump sum.
If the EAR is 8%, how much will you accept as a lump sum?

We can look at this question in different ways. As the question looks for the lump-sum you will accept right here right now, let us initially use the concept of present value.

First, find the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate (EAR) of 8%.

$(1+i)^{12} = 1.08$
$(1+i) = 1.08^{1/12} = 1.006434030$
i = rate of interest per month = 0.006434030 = 0.6434030%

You will accept a lump - sum x which satisfies the following criterion:
The lump-sum x = the sum of the present values of the regular payments

$$x = \frac{1000}{((1.08)^{1/12})^1} + \frac{1000}{((1.08)^{1/12})^2} + \frac{1000}{((1.08)^{1/12})^3} + \ldots\ldots + \frac{1000}{((1.08)^{1/12})^{239}} + \frac{1000}{((1.08)^{1/12})^{240}}$$

Multiplying both sides by $((1.08)^{1/12})^{240}$ (avoiding fractions in the S_n formula):

$$x((1.08)^{1/12})^{240} = 1000((1.08)^{1/12})^{239} + 1000((1.08)^{1/12})^{238} + \ldots\ldots + 1000((1.08)^{1/12})^1 + 1000 \quad *$$

Reversing the right hand side of this equation (also to avoid fractions in the S_n formula):

$$x((1.08)^{1/12})^{240} = 1000 + 1000((1.08)^{1/12})^1 + \ldots\ldots + 1000((1.08)^{1/12})^{238} + 1000((1.08)^{1/12})^{239}$$

The right hand side of this equation represents a geometric series with $a = 1000, r = 1.08^{1/12}$ and $n = 240$

$$x(1.08)^{1/12})^{240} = 1000\frac{((1.08)^{1/12})^{240} - 1)}{((1.08)^{1/12} - 1)}$$

$$x = €122,077.73$$

Assuming that payments start in one month's time, the rate is fixed, and there is zero risk, the present value of the annuity of €1000 per month for 20 years is €122,077.73 based on an EAR of 8%. This is the minimum lump - sum you will accept.

Using the amortisation formula to solve Sample question 4.

One could also use the amortisation formula on page 31 of the Formulae and Tables booklet.
Given that *A*= €1000 find *P*.

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} \Rightarrow P = A\frac{(1+i)^t - 1}{i(1+i)^t} = 1000\frac{\left(\left(1.08\right)^{\frac{1}{12}}\right)^{240} - 1)}{(0.006434030)\left((1.08)^{\frac{1}{12}}\right)^{240}} = €122,077.73$$

(When the lottery pays you €1000 each month for 20 years they are really getting a loan of €122,077.73 from you at 8% (APR) for 20 years.)

Alternative view - using the concept of future value

While the question is looking for the present value of an annuity of €1000 per month for 20 years we could also look at it in terms of future value.

Let's say you accept a lump sum of €*x*.

Then the €*x* deposited at 8% (EAR) for 20 years should yield the same final value as the **final or future value** of the 240 monthly payments of €1000 paid to you at the end of each month for 20 years and then invested by you each month at 8% (EAR). (This is for the purposes of working out the lump sum only, as it is unlikely that you would spend none of the money during the 20 years and invest it all.)

Future value (FV) of the lump sum = Future value (FV) of the annuity

The future value (FV) of the annuity of €1000 per month for 20 years paid at the end of each month:

$$=1000\left((1.08)^{\frac{1}{12}}\right)^{239} + 1000\left((1.08)^{\frac{1}{12}}\right)^{238} + 1000\left((1.08)^{\frac{1}{12}}\right)^{237} + ...+1000$$

(The FV of the first payment is $1000\left((1.08)^{\frac{1}{12}}\right)^{239}$ as it earns interest for 239 months.

The FV of the last payment paid at the end of the 240th month is €1000 as it earns no interest.)

Check using S_{240} of a geometric series that the future value of this annuity is €568,999.07.

Future value (FV) of the lump sum *x* is $x(1.08)^{20}$............(ii)

Equating (i) and (ii) leads to the equation below which is exactly the same as equation * on the previous page.

$$x(1.08)^{20} = 1000((1.08)^{1/12})^{239} + 1000((1.08)^{1/12})^{238} + ... + 1000((1.08)^{1/12})^{1} + 1000 \quad *$$

Hence, in agreement with the other methods:

$$x = €122,077.73$$

Sample question 5 LCHL

Looking for the present value of an annuity with a finite number of terms

Jack won a prize in a lottery. He has been given a choice of two options:

Option A: Receive an annuity of €1500, ***at the beginning*** of each month for 25 years.
Option B: Take a lump sum instead.

Jack decides to take Option **B.**
a) What lump sum should he accept assuming an AER of 4%?
b) He invests the lump sum he receives for 20 years in an account that pays 4% (AER).
How much will Jack's investment have amounted to after the 20 years?

As in Sample question 4 can look at this question in different ways. As the question looks for the lump-sum you will accept right here right now, let us initially use the concept of present value.
Note that the payments are made in this case at the beginning rather than at the end of each month.

First, find the rate of interest per month that would, if paid and compounded monthly, be equivalent to an effective annual rate (EAR) of 4%.

$(1+i)^{12} = 1.04$
$(1+i) = 1.04^{1/12} = 1.003273740$
i = rate of interest per month = 0.003273740 = 0.3273740%

You will accept a lump - sum x which satisfies the following criterion:
The lump-sum x = the sum of the present values of the regular payments

$$x = 1500 + \frac{1500}{((1.04)^{1/12})^1} + \frac{1500}{((1.04)^{1/12})^2} + \frac{1500}{((1.04)^{1/12})^3} + \dots\dots\dots + \frac{1500}{((1.04)^{1/12})^{289}} + \frac{1500}{((1.04)^{1/12})^{299}}$$

At this point, in Sample question 4, we avoided fractions. However this time we will omit this step and find Sn of a geometric series for the right hand side of the above equation as it is.

$$a = 1500, \quad r = \frac{1}{1.04^{\frac{1}{12}}}, \quad n = 300$$

$$S_{300} = a\frac{(1-r^n)}{(1-r)} = 1500\frac{\left(1-\left(\frac{1}{1.04^{\frac{1}{12}}}\right)^{300}\right)}{\left(1-\left(\frac{1}{1.04^{\frac{1}{12}}}\right)\right)} = €287,253.54$$

(Use of the memory facility on the calculator to store the reciprocal of $(1.04)^{1/12}$ is recommended here.)

Alternative view - using the concept of future value

a) The future value (FV) of the lump sum Jack accepts should be the same as the future value of the annuity.

The FV of an annuity of €1500 per month for 25 years paid ***at the beginning of*** each month is:

$$1500\left((1.04)^{\frac{1}{12}}\right)^{300} + 1500\left((1.04)^{\frac{1}{12}}\right)^{299} + 1500\left((1.04)^{\frac{1}{12}}\right)^{298} + \dots\dots\dots\dots + 1500\left((1.04)^{\frac{1}{12}}\right)^{1}$$

Writing this in reverse order:

FV of the annuity $= 1500(1.04^{1/12}) + 1500(1.04^{1/12})^2 + \dots\dots\dots 1500(1.04^{1/12})^{300}$

This is a geometric series with $a = 1500(1.04)^{1/12}, r = (1.04)^{1/12}$ and $n = 300$

FV of the annuity $= 1500(1.04)^{\frac{1}{12}} \dfrac{\left(\left(1.04\right)^{\frac{1}{12}}\right)^{300} - 1}{(1.04)^{\frac{1}{2}} - 1}$

FV of the lump sum x $= x(1.04)^{1/12})^{300} = x(1.04)^{25}$

FV of the lump sum x = FV of the annuity

$$x(1.04)^{25} = 1500(1.04)^{1/12} \frac{((1.04)^{1/12})^{300} - 1)}{(1.04)^{1/12} - 1}$$

$$x = \frac{€765770.9336}{2.665836331} = €287,253.54$$

Assuming that payments start at the beginning of each month, the rate is fixed, and there is zero risk the present value of the annuity of €1500 per month for 25 years is €287,253.54 based on an EAR of 4%. This is the amount of money that, if the lottery board invested it at 4% (AER), would produce annual payments of €1500.

Alternatively use the formula on page 31 of the tables. ($A = €1500,\ i = (1.04)^{1/12}$, find P)

You need to multiply the answer from the formula by $(1+i)$ as the formula assumes that the annuity is paid at the end and not the beginning of the payment period. Hence multiply P by $(1+i)$.

(See page 12)

b) Jack's investment of €287235.54 will have amounted to €287235.54$(1.04)^{20}$ =€629,407.88

Sample question 6 **Present value of an annuity with an infinite number of terms**
Suppose that you expect to receive a payment of €100 once per year, at the end of each year, for an indefinitely long time. What is the present value of this annuity?
(An annuity is a sum of money to be paid in regular intervals.)

Receiving €100 a year from now is worth less to you than an immediate €100, because you cannot invest the money until you receive it. In particular, the present value of a €100 one year in the future is $\frac{100}{1+i}$ where i is the annual equivalent rate of interest expressed as decimal or a fraction.

Similarly, a payment of €100 two years in the future has a present value of $\frac{100}{(1+i)^2}$.

Therefore, the present value of receiving €100 per year for an indefinitely long time can be expressed as an infinite series

$$\frac{100}{1+i}+\frac{100}{(1+i)^2}+\frac{100}{(1+i)^3}+\frac{100}{(1+i)^4}+..........$$

This is an infinite geometric series with $a=\frac{100}{1+i}$ and $r=\frac{1}{1+i}$

$$S_\infty=\frac{a}{1-r}=\frac{\frac{100}{1+i}}{1-\frac{1}{1+i}}=\frac{\frac{100}{1+i}}{\frac{1+i}{1+i}-\frac{1}{1+i}}=\frac{100}{1+i-1}=\frac{100}{i}$$ (See Tables on page 22 for the formula.)

For example, if the yearly interest rate is 10% (i= 0.10), then the annuity has a present value of €1000.

Sample question 7 LCHL Savings Bonds

The "New Horizons" computer company needs to raise money to expand. It issues a 10 year bond of €1,000 that pays €30 at the end of every six month period for the 10 years. If the current market annual interest rate is 7.12% (AER), with interest added at six-month intervals, what is the fair market value of the bond?

Questions on bonds provide a nice application of the concept of present value. We may know the value of the bond in x years time but are interested in what its value is right now.

The bond certificate in this question promises two things – an amount of €1,000 to be paid in 10 years and a half-yearly payment of €30 for ten years. Therefore, to find the fair market value of the bond, we need to find the present value of the lump sum of €1,000 we are to receive in 10 years' time, as well as the present value of the €30 half-yearly payments for the 10 years.

$$\text{Present value of €1000 due in 10 year's time } = \frac{1000}{(1.0712)^{10}} = €502.68 = P_1$$

Present value of regular payments of €30 $= P_2$

$$P_2 = \frac{30}{\left(1.0712^{\frac{1}{2}}\right)^1} + \frac{30}{\left(1.0712^{\frac{1}{2}}\right)^2} + \ldots\ldots\ldots\ldots\ldots\ldots + \frac{30}{\left(1.0712^{\frac{1}{2}}\right)^{20}}$$

$$P_2\left(1.0712^{\frac{1}{2}}\right)^{20} = 30\left(1.0712^{\frac{1}{2}}\right)^{19} + 30\left(1.0712^{\frac{1}{2}}\right)^{18} + \ldots\ldots\ldots\ldots\ldots 30\left(1.0712^{\frac{1}{2}}\right)^{1} + 30 \;\ldots\ldots **$$

Reversing the right hand side to avoid having r as a fraction:

$$P_2\left(1.0712^{\frac{1}{2}}\right)^{20} = 30 + 30\left(1.0712^{\frac{1}{2}}\right)^{1} + \ldots\ldots\ldots\ldots\ldots 30\left(1.0712^{\frac{1}{2}}\right)^{18} + 30\left(1.0712^{\frac{1}{2}}\right)^{19}$$

Right hand side is a geometric series with $a = 30,\ r = 1.0712^{\frac{1}{2}}, n = 20$

$$S_n = 30\frac{((1.0712)^{1/2})^{20} - 1)}{(1.0712)^{1/2} - 1}$$

$$P_2\left((1.0712)^{1/2}\right)^{20} = 30\frac{((1.0712)^{1/2})^{20} - 1)}{(1.0712)^{1/2} - 1}$$

$$P_2 = €426.42$$

The present value of the lump-sum of €1,000 due in 10 years	= €502.68
The present value of the €30 semi-annual payments for 10 years	= €426.42
Therefore, the fair market value of the bond	= €502.68 + €426.42 = €929.10

Use of the amortisation formula to find the value of P_2:

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} \Rightarrow P_2 = A\frac{(1+i)^t - 1}{i(1+i)^t} = 30\frac{\left(\left(1.0712^{\frac{1}{2}}\right)^{20} - 1\right)}{\left(1.0712^{\frac{1}{2}} - 1\right)\left(1.0712^{\frac{1}{2}}\right)^{20}} = €426.42$$

Interpreting ** on the previous page

To find the present value of the semi annual payments of €30 is the same as asking what lumpsum payment now would have the same future value if invested at 7.12% AER as an annuity of €30 paid semi-annually for 10 years given an AER of 7.12%.

The future value (FV) of the lump sum of €P_2 is given by

$$P_2\left((1.0712)^{\frac{1}{2}}\right)^{20} \qquad \text{(i)}$$

The future value (FV) of the annuity of €30 paid semi-annually for 10 years paid at the end of each 6 month period:

$$= 30\left((1.0712)^{\frac{1}{2}}\right)^{19} + 30\left((1.0712)^{\frac{1}{2}}\right)^{18} + 30\left((1.0712)^{\frac{1}{2}}\right)^{17} + ... + 30$$

Sample question 8 LCHL Pre-paying a loan i.e. paying off a loan early, before the loan's term is over i.e. paying off the unpaid balance

Mr. Mooney bought his house in 1975. He obtained a loan from the bank for 30 years at an interest rate of 9.8% APR. His monthly payment at the end of each month was €1260. In 1995, Mr. Mooney decided to pay off the loan.

Find the balance of the loan, i.e. the amount he owed, at that time.

Note: We do not know or need to know the original loan amount.

Since this is a question on loan repayments it is a good idea to look at the present value(s).

Mr. Mooney has made payments for 20 years (240 months) so he still has 120 payments to make so the bank should charge him the present value of these payments.

Method 1:

The amount owing on a loan at any time is the present value of the <u>remaining</u> repayments.

We can think in terms of getting the present value of 120 payments of €1260 due to be paid at the end of each month for 120 months.

$$x=\frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{1}}+\frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{2}}+\dots\dots\dots\frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{119}}+\frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{120}} \qquad \text{(i)}$$

The right hand side is a geometric series with $a=\dfrac{1260}{\left((1.098)^{\frac{1}{12}}\right)}$, $n=120$ and $r=\dfrac{1}{1.098^{\frac{1}{12}}}$

$$x=S_{120}=a\frac{\left(1-r^{n}\right)}{\left(1-r\right)}=\frac{1260}{\left((1.098)^{\frac{1}{12}}\right)}\frac{\left(1-\left(\dfrac{1}{1.098^{\frac{1}{12}}}\right)^{120}\right)}{\left(1-\left(\dfrac{1}{1.098^{\frac{1}{12}}}\right)\right)}$$

Evaluate $r=\dfrac{1}{1.098^{\frac{1}{12}}}=0.992239408$ and store it in the calculator's memory. Recall it each time it is needed in calculation.

$x=€97{,}847.55$

Alternative way of calculating *x* from equation (i) above:

$$x = \frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{1}} + \frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{2}} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{119}} + \frac{1260}{\left((1.098)^{\frac{1}{12}}\right)^{120}} \quad \text{(i)}$$

Multiply both sides by $\left((1.098)^{\frac{1}{12}}\right)^{120}$ to get:

$$x\left((1.098)^{\frac{1}{12}}\right)^{120} = 1260\left((1.098)^{\frac{1}{12}}\right)^{119} + 1260\left((1.098)^{\frac{1}{12}}\right)^{118} + 1260\left((1.098)^{\frac{1}{12}}\right)^{117} + \ldots\ldots + 1260 \;{**}$$

$$x\left((1.098)^{\frac{1}{12}}\right)^{120} = 1260 + 1260\left((1.098)^{\frac{1}{12}}\right)^{1} + 1260\left((1.098)^{\frac{1}{12}}\right)^{2} + \ldots\ldots\ldots\ldots\ldots\ldots + 1260\left((1.098)^{\frac{1}{12}}\right)^{119}$$

$$x\left((1.098)^{\frac{1}{12}}\right)^{120} = a\frac{\left(1-r^{n}\right)}{1-r} = 1260\frac{\left(1-\left((1.098)^{\frac{1}{12}}\right)^{120}\right)}{1-\left((1.098)^{\frac{1}{12}}\right)}$$

$$x = €97{,}847.55$$

** To rationalise this line, we could ask the question – what lump sum €*x* invested for 120 months will give the same final value as a regular investment of €1260 per month for 120 months?
<u>The future value (FV) of the lump sum</u> of €*x* is given by

$$x\left((1.098)^{\frac{1}{12}}\right)^{120} \quad \text{(ii)}$$

<u>The future value(FV) of the annuity of €1260 paid at the end of each month for 10 years</u>:

$$1260\left((1.098)^{\frac{1}{12}}\right)^{119} + 1260\left((1.098)^{\frac{1}{12}}\right)^{118} + 1260\left((1.098)^{\frac{1}{12}}\right)^{117} + \ldots\ldots + 1260 \;\text{(iii)}$$

Equation (ii) above is what the bank would earn if it invested lump sum €*x*, which is the present value of the annuity still owed to the bank. The right hand side shows how much the bank will get if Mr. Mooney continues to make the payments from now on for the next 10 years at the end of each month.

Equating these two future values (ii) and (iii) is what we have in ** above.

Using the amortisation formula for Sample Q8 LCHL Set A:

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} \Rightarrow P = A\frac{(1+i)^t - 1}{i(1+i)^t} = 1260\frac{\left((1.098)^{\frac{1}{12}}\right)^{120} - 1)}{((1.098)^{\frac{1}{12}} - 1)\left((1.098)^{\frac{1}{12}}\right)^{120}} = €97,847.55$$

Finding out the amount of money borrowed by Mr. Mooney

If you wished to, although it is not necessary for the above question, how would you calculate Mr. Mooney's loan?

Method 1

Loan = sum of the present value of all the repayments.

$$\text{Loan} = \frac{1260}{(1+i)^1} + \frac{1260}{(1+i)^2} + \frac{1260}{(1+i)^3} + \frac{1260}{(1+i)^4} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \frac{1260}{(1+i)^{359}} + \frac{1260}{(1+i)^{360}}$$

$$\text{Loan } (1+i)^{360} = 1260(1+i)^{359} + 1260(1+i)^{358} + \ldots\ldots\ldots + 1260(1+i)^1 + 1260$$

Reversing the order of the right hand side of the above equation -

$$\text{Loan } (1+i)^{360} = 1260 + 1260(1+i)^1 + \ldots\ldots\ldots + 1260(1+i)^{358} + 1260(1+i)^{359}$$

Right hand side is S_n of a geometric series with $a = 1260$ and $r = (1+i)$ and $n = 360$

$$\text{Loan } (1+i)^{360} = \frac{1260((1+i)^{360} - 1)}{1+i-1} = \frac{1260(1.007821290^{360} - 1)}{0.007821290} = 2500621.530$$

$$\text{Loan} = \frac{2500621.530}{1.007821290^{360}} = €151348.3620$$

Using the "amortisation –mortgages and loans" formula

$$A = P\frac{i(1+i)^t}{(1+i)^t - 1} \Rightarrow P = A\frac{(1+i)^t - 1}{i(1+i)^t} = 1260\frac{\left(\left((1.098)^{\frac{1}{12}}\right)^{360} - 1\right)}{\left((1.098)^{\frac{1}{12}} - 1\right)\left((1.098)^{\frac{1}{12}}\right)^{360}} = \frac{19558.08362}{0.1292256048} = €151,348.36$$

Method 2:

You can find the unpaid balance by looking at an amortisation schedule, which requires knowing how much was borrowed.

Method 3:

If an amortisation schedule is not available you can approximate the unpaid balance (quite accurately) as follows:

Unpaid balance = Current value of the loan amount – current value of the annuity

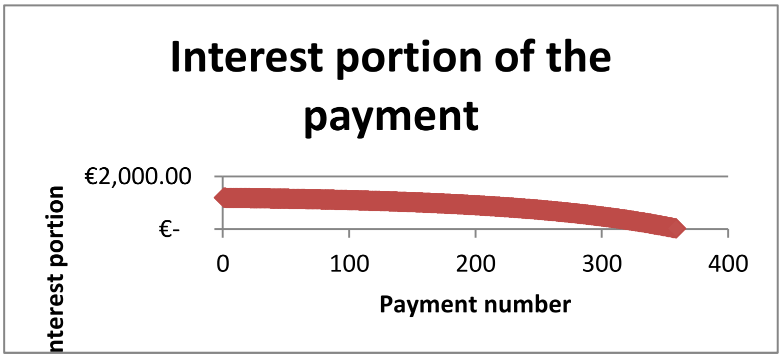
(In other words, if the bank hadn't given Mr Mooney the loan but had invested this amount instead, for 240 months, at the same interest rate as they are charging Mr. Mooney for the loan, what would be the difference between the current value of the invested "loan amount" and the value of the actual investment of €1260 every month from Mr. Mooney which they have invested at the same interest rate for the 240 months. This is what Mr. Mooney owes them to clear his debt.)

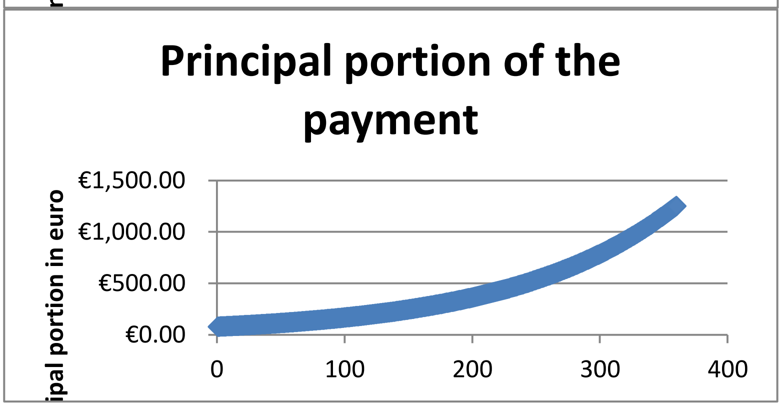
When a loan is paid off, the unpaid balance is zero and hence the current value of the loan amount equals the current value of the annuity.

Unpaid balance = current value of the loan amount - current value of the annuity

$=P(1+i)^n - pymt\dfrac{(1+i)^n - 1}{i}$ where P is the loan amount, $pymt$ is the loan payment, i is the periodic interest rate, n is the number of periods *from the beginning of the loan to the present*.

Unpaid balance $=151{,}348.36(1.007821290)^{240} - 1260\dfrac{(1.007821290)^{240} - 1}{0.007821290} = €97{,}847.55$





GROSS Before TAX		NTMA State Savings™ Products	NET After TAX	
Total	AER[1]		Total	AER[1]
10%	3.23%	3 year Savings Bonds[2]	10%	3.23%
15%	3.56%	4 year National Solidarity Bonds[5]	13.92%	3.31%
21%	3.53%	5½ year Savings Certificates[2]	21%	3.53%
50%	4.14%	10 year National Solidarity Bonds[5]	47.3%	3.95%

1.

(a) Show that the gross rate before tax (and after tax) of 10% in the 3 year savings bond referred to in this advertisement is equivalent to an AER (annual equivalent rate) of 3.23%.

(b) Show that the gross rate of 21% on a 5.5 year savings bond is equivalent to an AER of 3.53%.

2. Alice and Ken open an account with an interest rate of 3.4% AER. They deposit €1000 at the end of each year for 10 years. What is the future (final) value of their annuity? How much will they earn on their investment?

3. The management company of an apartment block estimates that they will need €30,000 in 4.5 years time to repaint the outside of the building and common areas.
If regular payments are made to a (sinking) fund earning 2.75% AER calculate

(i) The rate of interest per month, that would if paid and compounded monthly, be equivalent to an effective annual rate of 2.75%.
(ii) How much must be deposited in the fund at the end of each month to meet this target?

(iii) How much interest will be earned in the 4.5 years?

4. Grandparents Joe and Melissa want to start a regular savings account for their new grandchild so that on his 18^{th} birthday he will have €20,000 to help fund his education. How much will they deposit at the end of each month to achieve this target if they avail of a regular savings scheme at 2.5% AER?

5. Eddie plans to deposit €400 at the end of each month for 3 years in an account earning 3.25% AER. What single sum of money would Eddie need to invest now to achieve the same future /final value? What does your answer mean?

6. Jean won €1,000,000 in a lottery to be received in four annual payments of €250,000. She will receive the first payment in exactly one year from now. What is the present value of the four payments if the interest rate is 4.2% AER? Did it cost the lottery €1,000,000 to pay Jean her prize money? Explain.

7. Marian contributed €100 at the end of each week for 20 years to a pension (superannuation) fund earning 4.6% AER. Take 1 year to be 52 weeks.
(a) Find the rate of interest per week, which would if paid and compounded weekly, be equivalent to an effective annual rate of 4.6%.

(b) How much was her lump sum payment (to the nearest euro) when she retired?
(c) Find the rate of interest per month, which would if paid and compounded monthly, be equivalent to an effective annual rate of 3.8%.
(d) Marian used her lump sum to buy an annuity at 3.8% AER giving her an allowance at the end of each month for the next 20 years. How much is her monthly allowance to the nearest euro?

Loan repayments: A series of loan repayments is an annuity where the amount borrowed is the present value of the series of repayments. The amount owing on a loan at any time is the present value of the remaining repayments.

8. Ellen and Mike get a loan of €200,000 to be repaid at the end of each month in a series of equal payments over 25 years. The interest rate for the loan is 8.00% APR. Calculate:

   (i) the rate of interest per month, that would if paid and compounded monthly, be equivalent to an effective annual rate of 8.00%
   (ii) the amount of each monthly repayment
   (iii) the total amount to be repaid
   (iv) the total interest to be paid
   (v) If they had paid fortnightly, what would the repayment amount be and what would the total interest paid be in this case?

   (Possible extension question: How much would Ellen and Mike save by paying every fortnight instead of every month?)

9. A graduate paid €400 at the end of each month for five years to pay back a loan he borrowed while in College at an APR of 8.30%. How much did he borrow?

Also:

Q6 2011 LCHL Mathematics Paper 1 www.examinations.ie

Suggested solutions to the above questions

1.

(a) For every €1 you invest you get back €1.1 in 3 years time.

$$F = P(1+i)^t$$
$$1.1 = 1(1+i)^3$$

$$1+i = (1.1)^{\frac{1}{3}}$$
$$1+i = 1.03228...$$
$$i = 0.03228...$$

EAR = 3.23% (as advertised)

(b) For every €1 you invest you get back €1.21 in 5 years time.

$$F = P(1+i)^t$$
$$1.21 = 1(1+i)^{5.5} = (1+i)^{\frac{11}{2}}$$

$$1+i = (1.21)^{\frac{2}{11}}$$
$$1+i = 1.035265...$$
$$i = 0.035265...$$

EAR = 3.53% (as advertised)

2. Alice and Ken open an account with an interest rate of 3.4% AER. They deposit €1000 at the end of each year for 10 years. What is the future (final) value of their annuity? How much will they earn on their investment?
(You know the regular payment – you don't know the future value)

The sum of all the future values of all the regular savings of €1000 per month is

$$FV = 1000 + 1000(1.034)^1 + ... + 1000(1.034)^8 + 1000(1.034)^9$$

This is a geometric series with $a = 300, r = (1.0575)^{1/4}$ and $n = 16$

$$S_n = \frac{a(r^n - 1)}{r-1} = \frac{1000\left[(1.034)^{10} - 1\right]}{1.034 - 1} = €11,677.32$$

They earn €1677.32 on their investment.

3.

(i) For every €1 you invest you get back €1.0275 in 12 months.

$$F = P(1+i)^t$$
$$1.0275 = 1(1+i)^{12}$$

$$1+i = (1.0275)^{\frac{1}{12}}$$
$$1+i = 1.00226328...$$
$$i = 0.00226328...$$

(ii)

$$€30,000 = P(1+i)^{53} + P(1+i)^{52} + P(1+i)^{51} + ... + P(1+i)^2 + P(1+i)^1 + P$$
$$€30,000 = P + P(1+i)^1 + P(1+i)^2 + ... + P(1+i)^{52} + P(1+i)^{53}$$
$$€30,000 = S_{54} \text{ of a geometric series with } a = P,\ r = (1+i) = 1.00226328,\quad n = 54$$
$$€30,000 = S_{54} = P\frac{\left((1.00226328)^{54} - 1\right)}{0.00226328}$$
$$P = €522.93$$

(iii)

$$\text{Interest } = €30,000 - \left(€522.93\right)54 = €1761.78$$

4.

$$F = P(1+i)^t$$
$$1.025 = 1(1+i)^{12}$$

$$1+i = (1.025)^{\frac{1}{12}}$$
$$1+i = 1.002059836...$$
$$i = 0.002059836...$$

$$€20,000 = P(1+i)^{216} + P(1+i)^{215} + P(1+i)^{214} + ... + P(1+i)^2 + P(1+i)^1 + P$$
$$€20,000 = P + P(1+i)^1 + P(1+i)^2 + ... + P(1+i)^{215} + P(1+i)^{216}$$
$$€20,000 = S_{216} \text{ of a geometric series with } a = P,\ r = (1+i) = 1.002059836,\quad n = 216$$
$$€20,000 = S_{216} = P\frac{\left((1.002059836)^{216} - 1\right)}{0.002059836}$$
$$P = €73.61$$

5. Future value of the lump - sum deposited at 3.25% AER = future value of an annuity of €400 per month at 3.25% AER

$$F = P(1+i)^t$$
$$1.0325 = 1(1+i)^{12}$$

$$1+i = (1.0325)^{\frac{1}{12}}$$
$$1+i = 1.002668809...$$
$$i = 0.002668809...$$

$$€x(1.0325)^3 = 400(1+i)^{35} + 400(1+i)^{34} + 400(1+i)^{33} + ... + 400(1+i)^2 + 400(1+i)^1 + 400$$

i on the right hand side of this equation is the monthly rate of an EAR of 3.25%

$$€x(1.0325)^3 = 400 + 400(1+i)^1 + 400(1+i)^2 + ... + 400(1+i)^{33} + 400(1+i)^{34} + 400(1+i)^{35}$$

$$€x(1.0325)^3 = S_{36} \text{ of a geometric series with } a = P,\ r = (1+i) = 1.002668809,\ \ n = 36$$

$$€x(1.0325)^3 = S_{36} = 400\frac{\left((1.002668809)^{36} - 1\right)}{0.002668809}$$

$$€x(1.0325)^3 = S_{36} = €15093.33611$$
$$x = €13,712.45$$

Eddie would have to have €13,712.45 now to invest at 3.25% AER for three years to achieve the same final value as investing €400 per month for three years at 3.25% AER.

6.

Present value of the four payments of €250,000

$$= \frac{250,000}{(1+i)} + \frac{250,000}{(1+i)^2} + \frac{250,000}{(1+i)^3} + \frac{250,000}{(1+i)^4}$$

$$= S_4 \text{ of a geometric series with } a = \frac{250,000}{(1+i)}, r = \frac{1}{(1+i)}, n = 4, i = 0.042$$

$$= \frac{250,000}{(1+0.042)} \frac{\left(1 - \frac{1}{(1.042)}^4\right)}{\left(1 - \frac{1}{(1.042)}\right)} = €903,212.70$$

Paying in four instalments of €250,000 saved the lottery €96,787.30

Alternatively one could use the amortisation formula from Page 31 of the formula and tables booklet using P as the subject of the formula.

$$P = \frac{A\left((1+i)^t - 1\right)}{i(1+i)^t} = \frac{250000\left((1.042)^4 - 1\right)}{0.042(1.042)^4}$$

7.

(a)

$$F = P(1+i)^t$$
$$1.046 = 1(1+i)^{52}$$

$$1+i = (1.046)^{\frac{1}{52}}$$
$$1+i = 1.000865247$$
$$i = 0.000865247$$

(b)

Final (future) value $= 100(1+i)^{1039} + 100(1+i)^{1038} + 100(1+i)^{1037} + \dots\dots\dots\dots + 100(1+i)^1 + 100$

i on the right hand side of this equation is the weekly rate of an EAR of 4.6%

$$F = 100 + 100(1+i)^1 + 100(1+i)^2 + \dots\dots\dots\dots + 100(1+i)^{1037} + 100(1+i)^{1038} + 100(1+i)^{1039}$$

$F = S_{1040}$ of a geometric series with $a = 100,\ r = (1+i) = 1.000865247,\ \ n = 1040$

$$F = S_{1040} = 100\frac{\left((1.000865247)^{1040} - 1\right)}{0.000865247}$$
$$S_{1040} = €168,540.69$$

(c)

$$F = P(1+i)^t$$
$$1.038 = 1(1+i)^{12}$$

$$1+i = (1.038)^{\frac{1}{12}}$$
$$1+i = 1.003112817$$
$$i = 0.003112817$$

(d)

$$€168,540.69=\frac{A}{(1+i)^1}+\frac{A}{(1+i)^2}+\ldots\ldots\ldots\ldots\ldots\ldots\frac{A}{(1+i)^{239}}+\frac{A}{(1+i)^{240}}$$

Right hand side is S_n of a geometric series with $a=\frac{A}{(1+i)^1}, i=0.003112817,\ r=\frac{1}{(1+i)}$ and $n=240$

$$€168,540.69=\frac{A}{(1.003112817)^1}\frac{\left(1-\left(\frac{1}{(1.003112817)}\right)^{240}\right)}{(0.003112817)}$$

$A=€997.98=€998$ to the nearest euro

Alternatively use the amortisation and loans formula on page 31 of the formula and tables booklet

$$A=P\frac{i(1+i)^t}{(1+i)^t-1}=168,540.69\frac{0.003112817(1.003112817)^{240}}{(1.003112817)^{240}-1}=€997.98=€998 \text{ to the nearest euro}$$

8.

$$F=P(1+i)^t$$
$$1.08=1(1+i)^{12}$$

$$1+i=(1.08)^{\frac{1}{12}}$$
$$1+i=1.006434030$$
$$i=0.006434030$$

Using the amortisation and loans formula on page 31 of the formula and tables booklet

$$A=P\frac{i(1+i)^t}{(1+i)^t-1}=200000\frac{0.006434030(1.006434030)^{300}}{(1.006434030)^{300}-1}=€1506.83$$

(Alternatively use S_n of a geometric series.)

(iii) The total amount to be repaid $=300(€1506.83)=€452,049.00$

(iv) Interest paid $=€252,049.00$

9.

Loan = The sum of the present values of all the repayments

$$P=\frac{400}{(1+i)}+\frac{400}{(1+i)^2}+\frac{400}{(1+i)^3}+\ldots\ldots\ldots\ldots\ldots\frac{400}{(1+i)^{59}}+\frac{400}{(1+i)^{60}}$$

The right hand side is S_n of a geometric series with $a=\frac{400}{(1+i)}, r=\frac{1}{(1+i)}, i=(1.083)^{\frac{1}{12}}, n=60$

$$S_n=a\frac{(1-r^n)}{(1-r)}=\frac{400}{(1+i)}\frac{\left(1-\left(\frac{1}{(1+i)}\right)^n\right)}{\left(1-\frac{1}{(1+i)}\right)}=€19,727.35$$

Store $\frac{1}{(1+i)}$ in the calculator memory and recall as needed to facilitate calculation.

Alternatively use the amortisation and loan formula on page 31 of the tables and formula booklet to calculate P, given $A=€400,\ i=(1.083)^{\frac{1}{12}},\ \text{t}=60$

$$P=\frac{A\left((1+i)^t-1\right)}{i(1+i)^t}=€19,727.35$$

Continuous compounding and the number e

For a given **nominal rate** of interest (Page 32, formulae and tables booklet) the only factor that influences the final value is the number of compounding periods per year. To see the effect of different numbers of compounding periods per year we can look at the following situation.

Suppose you invest €1 for one year, at a nominal interest rate of 100% per year, compounded n times during the year.

What will be the final value after 1 year, if the interest is added:

i. at the end of the year
ii. every 6 months,
iii. every 3 months
iv. every month
v. every week, day , hour, minute, second

The results are shown in the table below:

How often interest is compounded	Final value
Yearly	$F = 1(1+1)^1 = 2$
Every 6 months	$F = 1(1+\frac{1}{2})^2 = 2.25$
Every 3 months	$F = 1(1+\frac{1}{4})^4 = 2.44140625$
Every month	$F = 1(1+\frac{1}{12})^{12} = 2.61303529$
Every week	$F = 1(1+\frac{1}{52})^{52} = 2.69259695$
Every day	$F = 1(1+\frac{1}{365})^{365} = 2.71456748$
Every hour	$F = 1(1+\frac{1}{8760})^{8760} = 2.71812669$
Every minute	$F = 1(1+\frac{1}{525600})^{525600} = 2.71827923$
Every second	$F = 1(1+\frac{1}{31536000})^{31536000} = 2.71828162$

Conclusion: The final value gets bigger and bigger but the rate at which it is growing slows down and seems to be getting closer and closer to some fixed value close to €2.7182.

In fact, the €1 will not grow to more than €2.72 during the year, regardless of how often the interest is compounded. As *n*, the number of compounding periods gets larger and larger, the "unrounded" amount is getting closer and closer to the number called *e*.

We say that the $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}=e$ represents the unrounded amount to which €1 would grow in a year if the nominal rate were 100% and the number n of compounding periods per year increased without limit; that is when the interest is compounded continuously.

What if we continuously compounded €*P* for *t* years at a nominal annual interest rate *i*

(*i* = annual interest rate expressed as a decimal or a fraction)?

In other words what is the $\lim_{n\to\infty} P\left(1+\frac{i}{n}\right)^{nt}$?

With a little help from algebra -

Let $m=\frac{n}{i}\Rightarrow\frac{1}{m}=\frac{n}{i}$ and $n=mi$

$$\lim_{n\to\infty} P\left(1+\frac{1}{m}\right)^{mit}=\lim_{n\to\infty} P\left(\left(1+\frac{1}{m}\right)^{m}\right)^{it}$$

For a fixed i, as $m=\frac{n}{i}$, $m\to\infty$ as n $\to\infty$.

Thus $\lim_{n\to\infty} P\left(1+\frac{1}{m}\right)^{mit}=\lim_{m\to\infty} P\left(\left(1+\frac{1}{m}\right)^{m}\right)^{it}=P\lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{it}=Pe^{it}$

$\boldsymbol{A(t)=Pe^{it}}$

–the continuously compounded interest formula

Appendix 1

Formulae for the spreadsheet for the amortisation schedule on page 15

A	B	C	D	E	F	G
Pymt #	**total payment**	**Interest portion**	**Principal portion**	**balance**		
0				10000	**rate**	0.06
1	=$G$4	=E2*$G$2	=B3-C3	=E2-D3	**years**	10
2	=$G$4	=E3*$G$2	=B4-C4	=E3-D4	**fixed payment per year**	2373.964004
3	=$G$4	=E4*$G$2	=B5-C5	=E4-D5		
4	=$G$4	=E5*$G$2	=B6-C6	=E5-D6		
5	=$G$4	=E6*$G$2	=B7-C7	=E6-D7		